LOCAL WAVE EFFECTS NEAR A FRONT ADVANCING INTO CALM WATER

Raymond Edwin Cunningham

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by

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ABSTRACT

A theory is developed which describes the local wave effects near a wave front advancing into calm water and is uniformly valid near the front. The waves generated by an oscillating vertical boundary (a wavemaker) and propagating out into an infinitely long channel are investigated both theoretically and experimentally. Comparison of the theoretical and experimental results indicates that the theory does predict the wave phenomena about the front to a good degree of accuracy.

Conclusions are drawn regarding the transient wave response to a wavemaker and compared to those of John W. Miles regarding the transient gravity wave response to an oscillating pressure. The transient features of the wave front considered are the width of the transient zone at a given time since the disturbance to the free surface of the fluid and the length of the transient period at a given distance from the disturbance. It is shown that the conclusions regarding transient gravity wave phenomena apply without qualification to the waves generated by both oscillating pressure sources and moving surfaces.

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NOMENCLATURE

δ - Dirac delta function

η - Wave height

g - Acceleration due to gravity

G - Green's function

Γ - Gamma function

h - depth of fluid

H - Heaviside step function

I_m - Imaginary part of a complex variable

R_e - Real part of a complex variable

s - Displacement of wavemaker paddle

S - Amplitude of displacement of wavemaker paddle at free surface

V - The ratio of distance to time (x/t)

v - Same as V above.

w - Frequency of oscillation of wavemaker paddle

INTRODUCTION

The object of this study is to develop an expression which describes the local wave effects near a wave front advancing into calm water that is uniformly valid about the front. Limiting forms of this expression as $t \to \infty$ or $x \to \infty$ were developed historically by Lamb[1] and by Cauchy and Poisson[2] respectively where t represents elapsed time after an initial disturbance to the free surface of the fluid and x the distance from the disturbance. Lamb's result describes the waves resulting from a steadily oscillating disturbance where the elapsed time since the disturbance was initiated is assumed to be so large that all transient features have disappeared and only the so-called steady state phenomena remains. The Cauchy-Poisson expression describes the initial propagation of waves out from a disturbance but does not account for the time dependent aspects of the disturbance. The exact formulation of these expressions depends on the nature of the disturbance, e.g., pressure variations or moving surface - and are widely available in the literature.

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[1]Lamb, H.,"On Deep Water Waves", *Proc. Lond. Math. Soc.*, (2), 2, 371, (1904).

[2]Cauchy, A., Poisson, S. D. "Memoire sur al theorie des ondes", *Mem. de l'Acad. Roy. des Sciences*, i. (1816).

Neither of the above theories attempt to describe the transient wave phenomena spanning their respective limits. This study develops a theory that predicts the waves in this transient zone and which upon passage to the appropriate limit yields the historical result. The waves generated by an oscillating vertical boundary (a wavemaker) and propagating out into an infinitely long channel are investigated both theoretically and experimentally. The channel is assumed to be infinitely wide so that the problem is two-dimensional.

During the problem formulation phase of this analysis, the attention of the author was directed to a paper by Miles[3], which considered the transient gravity wave response to an oscillating pressure. Although the mechanism by which transient waves are generated is different in Miles' approach, the qualitative features of the resulting waves should be similar to those generated by a wavemaker. In his paper Miles developed an expression predicting the transient waves that was uniformly valid about the wavefront. His solution method, though apparently consistent, was not immediately satisfying as it involved finding the asymptotic expansion of a double integral by applying the method of stationary phase twice in succession. It was felt that a

[3] Miles, J. W., "Transient Gravity Wave Response to an Oscillating Pressure", *J. Fluid Mech.*, 13, (1962), pp. 145-150.

more direct means of obtaining a uniformly valid expression was desirable - both to verify Miles' result and to obtain clearer insight into the transient problem.

This study was undertaken primarily in the interest of academic completeness; viz., to bridge the gap between the solutions of Lamb and Cauchy-Poisson. However, several engineering applications can be foreseen. The utilization of wavemakers in towing tanks with corresponding applications such as ship model and ocean structure testing is of direct interest to the naval architect or ocean engineer. Transient features of generated waves may allow some aspects of model testing to be explored more rigorously and completely. Further application is found in the area of ocean wave forecasting. The wave generating models used presently in predicting the wave phenomena associated with a storm at sea are not completely satisfactory and a new model, based partially or entirely on the expression derived in this study could be formulated.

THEORETICAL ANALYSIS

Consider the waves generated by a vertical plate in oscillatory motion given by (1) in a fluid bounded by a free surface, the vertical plate, and of infinite extent otherwise. This so-called wavemaker problem is solved in Appendix B and the resulting expression for the waveheight as a function of distance and time is given by (2).

$$(1) \qquad s = S\, e^{w^2 y/g} \sin(wt)\, H(t)$$

where S = amplitude of displacement at the free surface

$H(t)$ is the Heaviside step function

$$(2) \qquad \eta(x,t) = \frac{wS}{\pi} \int_0^\infty \frac{\cos(ux)}{u + w^2/g} \left[\frac{\sin(wt) - \sin(\sqrt{gu}\, t)}{w - \sqrt{gu}} + \right.$$

$$\left. + \frac{\sin(wt) + \sin(\sqrt{gu}\, t)}{w + \sqrt{gu}} \right] du$$

Introducing the change in variable $u = \sigma^2/g$

$$(3) \quad \eta(x,t) = \frac{2wS}{\pi} \int_0^\infty \frac{\sigma \cos[(\sigma^2/g)x]}{\sigma^2 + w^2} \left[\frac{\sin(wt) - \sin(\sigma t)}{w - \sigma} + \right.$$

$$\left. + \frac{\sin(wt) + \sin(\sigma t)}{w + \sigma} \right] d\sigma =$$

$$= I_m \left\{ \frac{2wS}{\pi} \int_0^\infty \frac{\sigma^2 e^{i(\frac{\sigma^2}{g}x + \sigma t)} - \sigma^2 e^{i(\frac{\sigma^2}{g}x - \sigma t)} - w\sigma e^{i(\frac{\sigma^2}{g}x + wt)} + w\sigma e^{i(\frac{\sigma^2}{g}x - wt)}}{(\sigma^2 + w^2)\,(\sigma + w)\,(\sigma - w)} d\sigma \right\}$$

Using nondimensional variables $x = w^2 x/g$, $t = wt$, $\sigma = \sigma/w$ and $\eta = \frac{\pi\eta}{2S}$, (3) reduces to

(4)
$$\eta(x,t) =$$
$$= I_m \left\{ \int_0^\infty \frac{\sigma^2 e^{i(\sigma^2 x+\sigma t)} - \sigma^2 e^{i(\sigma^2 x-\sigma t)} - \sigma e^{i(\sigma^2 x+t)} + \sigma e^{i(\sigma^2 x-t)}}{(\sigma^2+1)\ (\sigma+1)\ (\sigma-1)} d\sigma \right\}$$

Equation (4) for $\eta(x,t)$ may be considered as the sum of four integrals.

(5)
$$\eta(x,t) = I_m \ \{I_1 - I_2 - I_3 + I_4\}$$

where
$$I_1 = \int_0^\infty f(\sigma)\ e^{i(\sigma^2 x+\sigma t)} d\sigma$$
$$I_2 = \int_0^\infty f(\sigma)\ e^{i(\sigma^2 x-\sigma t)} d\sigma$$
$$I_3 = \int_0^\infty \frac{f(\sigma)}{\sigma}\ e^{i(\sigma^2 x+t)} d\sigma$$
$$I_4 = \int_0^\infty \frac{f(\sigma)}{\sigma}\ e^{i(\sigma^2 x-t)} d\sigma$$
and
$$f(\sigma) = \frac{\sigma^2}{\sigma^4-1}$$

The asymptotic behavior of these integrals for large x or t is readily evaluated by treating σ as a complex variable and employing the residue theorem and Cauchy's integral theorem of complex variable theory. Their behavior for small x or t is given in Appendix H. In the complex σ plane, the real integrals in (5) are transformed to complex integrals which are to be

evaluated along the contour consisting of the ray from the origin to infinity along the positive real axis.

It is easily shown (see Appendix C) that the asymptotic behavior of integrals I_1, I_3 and I_4 is given by the following expressions for large t or x:

$$I_1 \sim \frac{i\pi}{4} e^{i(x+t)} + 0(t^{-3})$$

$$I_3 \sim \frac{i\pi}{4} e^{i(x+t)} + 0(t^{-2})$$

$$I_4 \sim \frac{i\pi}{4} e^{i(x-t)} + 0(t^{-2})$$

The asymptotic behavior of (5) then reduces to the following:

$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{4} e^{i(x-t)} - I_2 \right\} \tag{6}$$

The asymptotic behavior of I_2 is not as easily ascertained as was that of I_1, I_3 and I_4. In the evaluation of these integrals, the best choice of an equivalent contour to facilitate the determination of their asymptotic behavior was obvious. If a similar contour is chosen for the evaluation of I_2, however, the integral diverges along portions of the contour. Consider then the exponential term of I_2 in expanded form where the substitution $\sigma = r\ e^{i\theta}$ has been made. It takes the form

$$(7) \qquad e^{i(\sigma^2 x - \sigma t)} = e^{it\{r^2(x/t)\cos(2\theta) - r\sin(\theta)\}} \times$$

$$\times\ e^{-t\{r^2(x/t)\sin(2\theta) - r\sin(\theta)\}}$$

A good choice for an equivalent contour would be one on which this term becomes small as x and t become large and vanishes when the contour is extended to infinity. For this term to become exponentially small for large values of t (or x), the following restriction on r and θ applies:

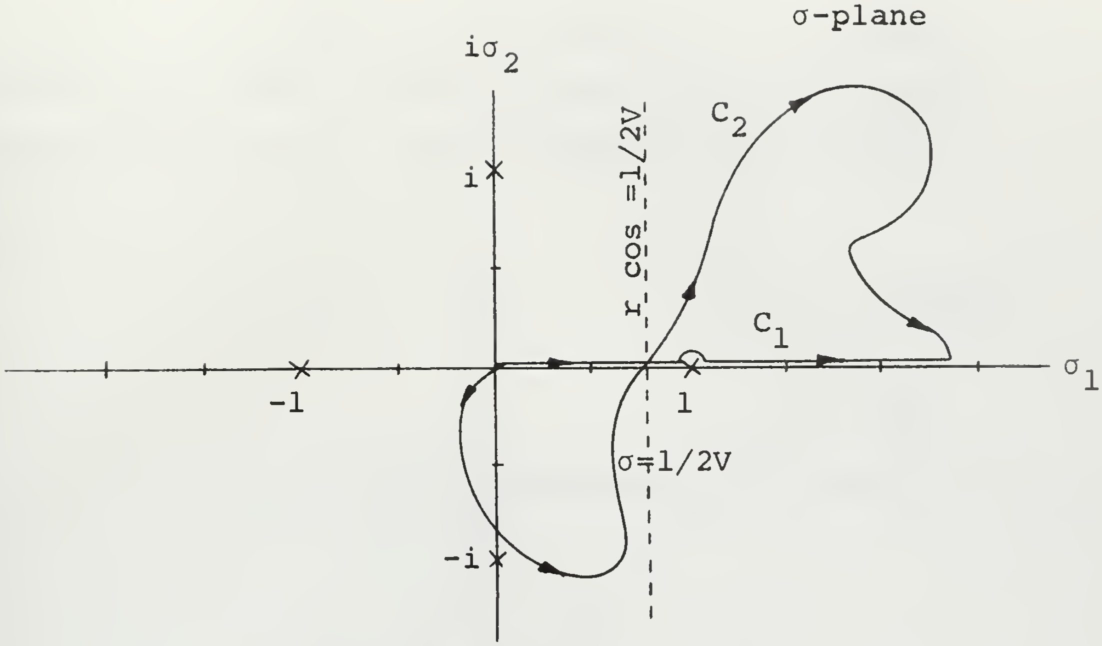
$$(8) \qquad r^2(x/t)\sin(2\theta) - r\sin(\theta) > 0$$

Let $x/t = V$: then for θ negative $r < 1/2V\cos(\theta)$

for θ positive $r > 1/2V\cos(\theta)$

for $\theta = 0$ $r = 1/2V$

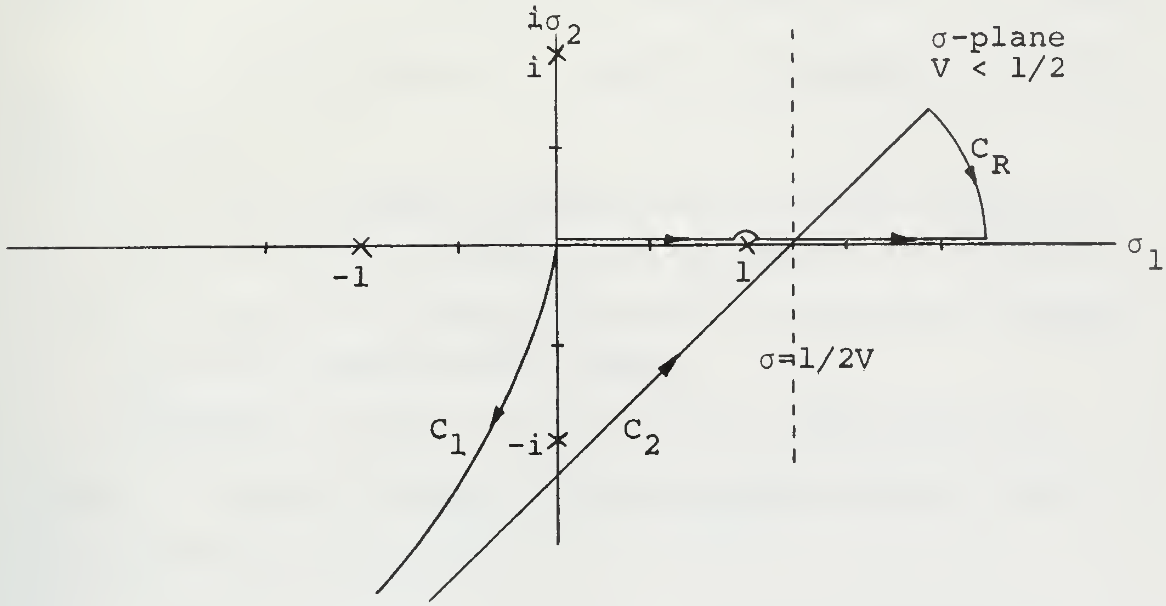
Thus, any contour with the general features of C_2 shown below is a satisfactory equivalent contour for the path from 0 to $+\infty$ along the real axis. The point $\sigma = 1/2V + 0i$ where the contour C_2 crosses the real axis is a stationary point of the integrand and must correspond to a saddle point as the integrand is a function of the complex variable σ and therefore satisfies Laplace's equation.

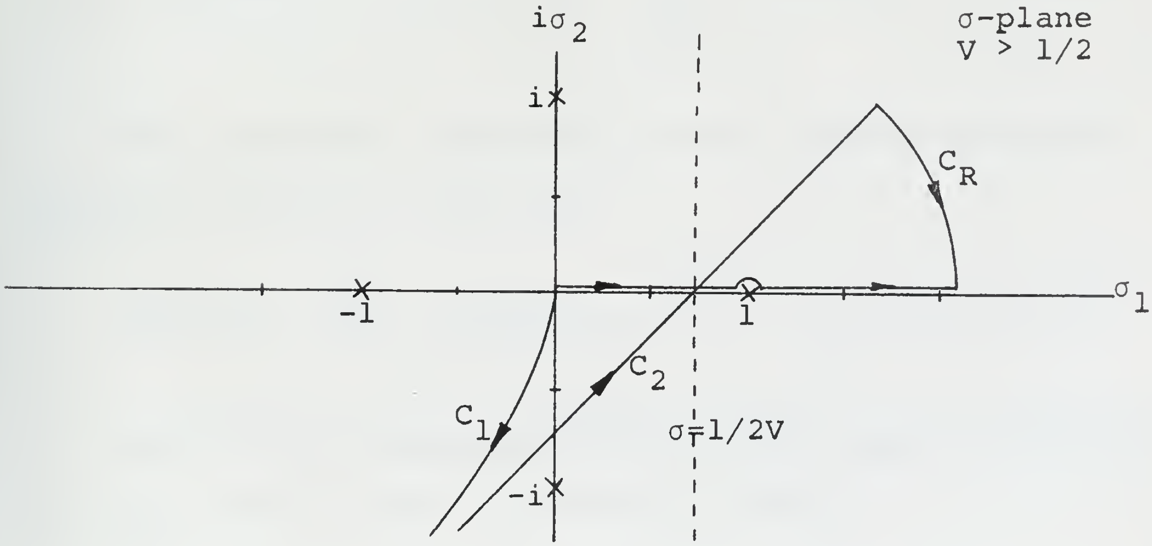


Here singularities of the integrand of I_2 have been indicated with x marks and in all cases are simple poles. The contour along the real axis is denoted by C_1 and has been indented at $\sigma = 1$. Thus, the above integral is defined in the Cauchy principle value sense. If the integral exists in the conventional sense, then its value will necessarily agree with the Cauchy principle value so there is no loss in generality.

The existence of a saddle point at $\sigma = 1/2V$ suggests the most convenient equivalent contour on which to evaluate I_2. If C_2 is deformed so that it coincides with the steepest descents path from the saddle point, then divergent

integrals are avoided as the integrand of I_2 becomes exponentially small on such a path (granting suitable restrictions on $f(\sigma)$). This contour is shown below.





Portions of C_2 that join the steepest descents path to the origin and to the positive real axis at infinity have been labeled C_1 and C_R. The location of the saddle point relative to the simple pole at $\sigma = 1/2V$ depends on the parameter $V = x/t$. Therefore, two general cases are shown.

The closing contours C_1 and C_R are chosen in the following way:

1) C_1 is the steepest descents path from the origin to infinity and intersects the contour C_2 at infinity.
2) C_R is the arc of the circle $Re^{i\theta}$ joining the contour C_2 to the positive real axis.

Consider the case $V = x/t < 1/2$. Application of Cauchy's integral theorem and the residue theorem lead to the following expression:

$$(9)\quad I_2 - i\pi \mathrm{Res}(1) = \int_{C_1} + \int_{C_2} + \int_{\substack{C_R \\ R\to\infty}} - 2\pi i \mathrm{Res}(1) - 2\pi i \mathrm{Res}(-i)$$

The asymptotic properties of each term in the above equality can now be evaluated.

$$\mathrm{Res}(-i) = \lim_{\sigma\to -i} \left[\frac{(\sigma+i)\sigma^2}{\sigma^4 - 1} e^{it(\sigma^2 V-\sigma)} \right] = \frac{i}{4} e^{-t} e^{-itV}$$

This term is exponentially small and can be neglected in comparison to terms algebraically small.

$$\mathrm{Res}(1) = \lim_{\sigma\to+1} \left[\frac{(\sigma-1)\sigma^2}{\sigma^4-1} e^{it(\sigma^2V-\sigma)} \right] = \frac{e^{i(x-t)}}{4}$$

$$\int_{C_R} \to 0 \text{ as } R \to \infty$$

It is easily shown (see Appendix C) that $\int_{C_1} = 0(t^{-3})$ and is therefore small compared with other contributions to I_2. Therefore, (9) reduces to

(10) $$I_2 = \int_{C_2} - \frac{i\pi}{4} e^{i(x-t)} + 0(t^{-3})$$

Consider

(11) $$\int_{C_2} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2V-\sigma)} d\sigma$$

where V = x/t is a parameter to be specified and is less than 1/2 for the case under consideration. Introducing the change in variable $-\frac{1}{4V} + i\tau = \sigma^2 V - \sigma$, the following considerations apply:

$$-\frac{1}{4V} + i\tau = \sigma^2 V - \sigma$$

$$\sigma = \frac{1}{2V} [1 \pm \sqrt{4iV}\, \tau^{1/2}] = \frac{1}{2V} [1 \pm \gamma\tau^{1/2}], \; \gamma = \sqrt{4iV}$$

$$\frac{d\sigma}{d\tau} = \frac{\pm \gamma}{4V\tau^{1/2}}$$

On C_2 in the upper half plane and in the half plane to the right of C_2

$$\sigma = \frac{1}{2V}\left[\, 1 + \gamma\tau^{1/2} \,\right] = \sigma_+ \;, \quad \frac{d\sigma_+}{d\tau} = \frac{\gamma}{4V\tau^{1/2}}$$

On C_2 in the lower half plane and in the half plane to the left of C_2

$$\sigma = \frac{1}{2V}\left[\, 1 - \gamma\tau^{1/2} \,\right] = \sigma_- \;, \quad \frac{d\sigma_-}{d\tau} = \frac{-\gamma}{4V\tau^{1/2}}$$

Therefore, the integral along C_2 is given by

$$(12) \quad \int_{C_2} \frac{\sigma^2}{\sigma^4 - 1}\, e^{it\{\sigma^2 (x/t) - \sigma\}} d\sigma = \int_{C_2} \frac{\sigma^2}{\sigma^4 - 1}\, e^{it(\sigma^2 V - \sigma)} d\sigma$$

$$= e^{\frac{-it}{4V}} \left[\int_0^\infty \frac{\sigma_+^2}{\sigma_+^4 - 1} \frac{d\sigma_+}{d\tau} e^{-t\tau} d\tau + \int_\infty^0 \frac{\sigma_-^2}{\sigma_-^4 - 1} \frac{d\sigma_-}{d\tau} e^{-t\tau} d\tau \right]$$

If the Laurent expansion about $\tau = 0$ is obtained, the above integrals may be evaluated by integrating term by term. It is more convenient, however, to subtract off the pole at $\sigma = 1$ and to then expand the remainder of the expression about $\tau = 0$ so that the effect of the pole may be considered separately with a view towards obtaining a uniformly valid result. These expansions are developed in Appendix D and are given below.

(13)
$$\frac{\sigma_+^2}{\sigma_+^4-1}\frac{d\sigma_+}{d\tau} = \frac{-\gamma/8}{1-2V}\left[\tau^{1/2} + \frac{1-2V}{\gamma}\right]^{-1} + \frac{\gamma}{8}\left[\frac{\tau^{-1/2}}{1-2V} + \right.$$

$$+ \frac{(-)\{1-4V-4V^2\}\tau^{-1/2}}{(1+2V)(1+4V^2)} + \frac{\gamma\{(1+4V^2)^2-8V(1+2V)^2\}}{(1+2V)^2(1+4V^2)^2} +$$

$$\left. + \frac{(-)\gamma^2\{(1+4V^2)^3-4V(1+2V)^3(3-4V^2)\}\tau^{1/2}}{(1+2V)^3\ (1+4V^2)^3} + \ldots\right]$$

(14)
$$\frac{\sigma_-^2}{\sigma_-^4-1}\frac{d\sigma_-}{d\tau} = \frac{\gamma/8}{1-2V}\left[\tau^{1/2} - \frac{1-2V}{\gamma}\right]^{-1} + \frac{\gamma}{8}\left[\frac{-\tau^{-1/2}}{1-2V} + \right.$$

$$+ \frac{\{1-4V-4V^2\}}{(1+2V)(1+4V^2)}\tau^{-1/2} + \frac{\gamma\{(1+4V^2)^2-8V(1+2V)^2\}}{(1+2V)^2(1+4V^2)^2} +$$

$$\left. + \frac{\gamma^2\{(1+4V^2)^3-4V(1+2V)^3(3-4V^2)\}}{(1+2V)^3\ (1+4V^2)}\tau^{1/2} + \ldots\right]$$

Noting that terms containing integral powers of τ cancel identically upon substitution into (12), the resulting expression becomes

(15)
$$\int_{C_2} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2 V-\sigma)} d\sigma = \frac{-\gamma/8}{1-2V} e^{-it/4V} \left[\int_0^\infty \frac{e^{-t\tau} d\tau}{\tau^{1/2} + \frac{1-2V}{\gamma}} + \right.$$

$$\left. + \int_0^\infty \frac{e^{-t\tau} d\tau}{\tau^{1/2} - \frac{1-2V}{\gamma}} \right] + e^{\frac{-it}{4V}} \left[2\int_0^\infty \left\{ \frac{\gamma/8 \ \tau^{-1/2}}{1-2V} - \frac{\gamma/8[1-4V-4V^2]}{(1+2V)(1+4V^2)} + \right. \right.$$

$$\left. \left. - \frac{\gamma^3/8[(1+4V^2)^3 - 4V(1+2V)^3(3-4V^2)]}{(1+2V)^3(1+4V^2)^3} \tau^{1/2} + O(\tau^{3/2}) \right\} e^{-t\tau} d\tau \right]$$

The first bracketed term in (15) includes the effect of the pole at $\sigma = 1$. The second bracketed term can be expressed as a series of Gamma functions and is not influenced by the pole at $\sigma = 1$. Introducing the change of variable $\tau^{1/2} = s$, the integrals in the first bracketed term above are transformed to the following:

(16)
$$\int_0^\infty \frac{e^{-t\tau} d\tau}{\tau^{1/2} + \frac{1-2V}{\gamma}} + \int_0^\infty \frac{e^{-t\tau} d\tau}{\tau^{1/2} - \frac{1-2V}{\gamma}} =$$

$$= \int_0^\infty \frac{e^{-ts^2} 2sds}{s + \frac{1-2V}{\gamma}} + \int_0^\infty \frac{e^{-ts^2} 2sds}{s - \frac{1-2V}{\gamma}}$$

Let $s = -s$ in the second integral of (16).

Then

(17) $$\int_0^\infty \frac{e^{-ts^2}2sds}{s+\frac{1-2V}{\gamma}} + \int_0^\infty \frac{e^{-ts^2}2sds}{s-\frac{1-2V}{\gamma}} = \int_0^\infty \frac{e^{-ts^2}2sds}{s+\frac{1-2V}{\gamma}} + \int_{-\infty}^0 \frac{e^{-ts^2}2sds}{s+\frac{1-2V}{\gamma}} =$$

$$= \int_{-\infty}^\infty \frac{e^{-ts^2}2sds}{s+\frac{1-2V}{\gamma}} = 2\int_{-\infty}^\infty \frac{(s+\frac{1-2V}{\gamma}-\frac{1-2V}{\gamma})e^{-ts^2}ds}{s+\frac{1-2V}{\gamma}} =$$

$$= 2\left[\int_{-\infty}^\infty e^{-ts^2}ds - \frac{1-2V}{\gamma}\int_{-\infty}^\infty \frac{e^{-ts^2}ds}{s+\frac{1-2V}{\gamma}}\right]$$

With the change in variable $y = t^{1/2}s$ the first integral in (17) is readily evaluated and the second placed in a more convenient form. Equation (17) then reduces to

(18) $$2\left[\sqrt{\pi}\, t^{-1/2} + \frac{\pi}{i}\left(\frac{1-2V}{\gamma}\right) W(z)\right]$$

where $$W(z) = \frac{i}{\pi}\int_{-\infty}^\infty \frac{e^{-y^2}dy}{z-y} \qquad [I_m(z) > 0]$$

$$z = -t^{1/2}\left(\frac{1-2V}{\gamma}\right)$$

$W(z)$ can be represented by the following expression[4]:

$$W(z) = e^{-z^2}\operatorname{erfc}(-iz)$$

where $$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt$$

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[4] Abramowitz, M. and Stegun, I. A. ed., *Handbook of Math. Functions*, New York, (1965), p. 297.

On substituting for W(z), (18) can be placed in the form

(20)
$$2\left[\sqrt{\pi}\, t^{-1/2} + \frac{\pi}{i}\left(\frac{1-2V}{\gamma}\right) W(z)\right] =$$

$$= 2\left[\sqrt{\pi}\, t^{-1/2} - i\pi\left(\frac{1-2V}{\gamma}\right) e^{-\left(\frac{1-2V}{\gamma}\right)^2 t} \operatorname{erfc}\left\{i\left(\frac{1-2V}{\gamma}\right) t^{1/2}\right\}\right]$$

Therefore, (11) can be expressed as follows for the case $V = x/t < 1/2$:

(21)
$$\int_{C_2} \frac{\sigma^2}{\sigma^4-1}\, e^{it(\sigma^2 V-\sigma)}\, d\sigma \sim$$

$$\frac{-\gamma/8}{1-2V}\, e^{\frac{-it}{4V}}\left[2\sqrt{\pi}t^{-1/2} - 2\pi i\left(\frac{1-2V}{\gamma}\right) e^{-\left(\frac{1-2V}{\gamma}\right)^2 t} \operatorname{erfc}\left\{i\left(\frac{1-2V}{\gamma}\right) t^{1/2}\right\}\right] +$$

$$+ e^{-it/4V}\left[\frac{\gamma/4}{1-2V}\,\Gamma\left(\frac{1}{2}\right) t^{-1/2} - \frac{\gamma/4\,\Gamma(1/2)\ \{1-4V-4V^2\} t^{-1/2}}{(1+2V)(1+4V)} +\right.$$

$$\left. + \frac{-\gamma^3/4\{(1+4V^2)^3 - 4V(1+2V)^3(3-4V^2)\}\Gamma(3/2)\, t^{-3/2}}{(1+2V)^3(1+4V^2)^3} + O(t^{-5/2})\right]$$

Substitution of (21) into (10) leads to the following expression for (6):

(22)

$$\eta(x,t) \sim I_m\left\{\frac{i\pi}{2} e^{i(x-t)} - \frac{i\pi}{4} e^{\frac{-it}{4V}} e^{-(\frac{1-2V}{\gamma})^2 t} \operatorname{erfc}\{i(\frac{1-2V}{\gamma}) t^{1/2}\} + \right.$$

$$+ e^{-i(\frac{t^2}{4x} - \frac{\pi}{4})} \left[\frac{\frac{V^{1/2}}{2}\{1-4V-4V^2\}}{(1+2V)(1+4V^2)} \Gamma(1/2) t^{-1/2} + \right.$$

$$\left.\left. + 2iV^{3/2} \frac{\{(1+4V^2)^3 - 4V(1+2V)^3(3-4V^2)\}}{(1+2V)^3(1+4V^2)^3} \Gamma(\tfrac{3}{2}) t^{-3/2} + 0(t^{-5/2}) \right]\right\} \quad (V<\tfrac{1}{2})$$

where $\gamma = \sqrt{4iV} = \sqrt{4i\ x/t}$

Consider the case $V = x/t > 1/2$. Application of Cauchy's integral theorem and the residue theorem leads to

(23) $$I_2 - i\pi \operatorname{Res}(1) = \int_{C_1} + \int_{C_2} + \int_{\substack{C_R \\ R\to\infty}}$$

Similar considerations regarding $\int_{C_1}$ and $\int_{C_R}$ apply as for the case $V = x/t < 1/2$. Therefore, the asymptotic behavior of I_2 for large t or x is given by

(24) $$I_2 = \int_{C_2} \frac{\sigma^2}{\sigma^4 - 1} e^{it(\sigma^2 V - \sigma)} d\sigma + \frac{i\pi}{4} e^{i(x-t)} + 0(t^{-3})$$

Proceeding as for the case $V < 1/2$, introduce the change in variable $-\frac{1}{4V} + i\tau = \sigma^2 V - \sigma$. All previous conclusions regarding the branches and branch cuts of $\sigma(\tau)$ apply, hence

(25)

$$\int_{C_2} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2 V-\sigma)} d\sigma =$$

$$= e^{\frac{-it}{4V}} \left[\int_0^\infty \frac{\sigma_+^2}{\sigma_+^4-1} \frac{d\sigma_+}{d\tau} e^{-t\tau} d\tau + \int_0^\infty \frac{\sigma_-^2}{\sigma_-^4-1} \frac{d\sigma_-}{d\tau} e^{-t\tau} d\tau \right]$$

The Laurent expansions previously obtained about $\tau = 0$ are valid, hence

(26)

$$\int_{C_2} \sim e^{\frac{-it}{4V}} \left[\frac{\gamma/4 \ \Gamma(1/2) t^{-1/2}}{1-2V} - \frac{\gamma/4\{1-4V-4V^2\}\Gamma(1/2) \ t^{-1/2}}{(1+2V)(1+4V^2)} - \right.$$

$$\left. - \frac{\gamma^{3}/4\{(1+4V^2)^3-4V(1+2V)^3(3-4V^2)\}}{(1+2V)^3(1+4V^2)^3} \Gamma(3/2) t^{-3/2} + 0(t^{-5/2}) \right] -$$

$$- \frac{\gamma/8}{1-2V} e^{\frac{-it}{4V}} \left[\int_0^\infty \frac{e^{-ts^2} 2sds}{s + \frac{1-2V}{\gamma}} + \int_0^\infty \frac{e^{-ts^2} 2sds}{s - \frac{1-2V}{\gamma}} \right]$$

The remaining integrals are treated in a similar manner as before except that the substitution $s = -s$ should be made in the first integral in lieu of the second integral of (26) so that the resulting formulation exists.

(27)
$$2\left[\int_0^{\infty}\frac{e^{-ts^2}sds}{s+\frac{1-2V}{\gamma}}+\int_0^{\infty}\frac{e^{-ts^2}sds}{s-\frac{1-2V}{\gamma}}\right]=$$

$$=2\left[\int_{-\infty}^{0}\frac{e^{-ts^2}sds}{s-\frac{1-2V}{\gamma}}+\int_0^{\infty}\frac{e^{-ts^2}sds}{s-\frac{1-2V}{\gamma}}\right]=2\int_{-\infty}^{\infty}\frac{e^{-ts^2}sds}{s-\frac{1-2V}{\gamma}}=$$

$$=2\left[\int_{-\infty}^{\infty}e^{-ts^2}ds+\frac{1-2V}{\gamma}\int_{-\infty}^{\infty}\frac{e^{-ts^2}ds}{s-\frac{1-2V}{\gamma}}\right]$$

Let $y = t^{1/2}s$.

$$=2\left[\sqrt{\pi}\,t^{-1/2}-\frac{\pi}{i}\left(\frac{1-2V}{\gamma}\right)W(z)\right]$$

where $z = t^{1/2}\left(\frac{1-2V}{\gamma}\right)$

(28)
$$=2\left[\sqrt{\pi}\,t^{-1/2}+i\pi\left(\frac{1-2V}{\gamma}\right)e^{-\left(\frac{1-2V}{\gamma}\right)^2 t}\operatorname{erfc}\left\{-i\left(\frac{1-2V}{\gamma}\right)t^{1/2}\right\}\right]$$

The evaluation of (25) is now complete and the wave-height $\eta(x,t)$ is therefore given by

(29)
$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{4}\, e^{\frac{-it^2}{4x}}\, e^{-\left(\frac{1-2V}{\gamma}\right)^2 t} \operatorname{erfc}\{-i(\tfrac{1-2V}{\gamma})t^{1/2}\} + \right.$$

$$+ e^{-i\left(\frac{t^2}{4x} - \frac{\pi}{4}\right)} \left[\frac{V^{1/2}/2\{1-4V-4V^2\}\Gamma(1/2)}{(1+2V)(1+4V^2)}\, t^{-1/2} + \right.$$

$$+ \frac{2iV^{3/2}\{(1+4V^2)^3 - 4V(1+2V)^3(3-4V^2)\}}{(1+2V)^3\ (1+4V^2)^3}\, \Gamma(\tfrac{3}{2})\, t^{-3/2} +$$

$$\left. \left. + 0(t^{-5/2}) \right] \right\} \qquad (V > 1/2)$$

where $\gamma = \sqrt{4iV}$

The asymptotic expansions for $\eta(x,t)$ have been derived for the two separate cases of $V = x/t < 1/2$ and $V = x/t > 1/2$. That these two expansions are equivalent can be shown by exploiting either of the following symmetry relations:

(30)
$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$$

(31)
$$w(-z) = 2e^{-z^2} - w(z)$$

where $w(z) = i/\pi \int_{-\infty}^{+\infty} e^{-t^2}/(z-t)\,dt \quad (I_m\ z > 0)$

Considering equations (22) and (29) for $\eta(x,t)$, symmetry relation (30) is most convenient.

From (30),

(32)
$$\operatorname{erfc}\{i(\frac{1-2V}{\gamma})t^{1/2}\} = 2 - \operatorname{erfc}\{-i(\frac{1-2V}{\gamma})t^{1/2}\}$$

Substituting (32) into (22) one obtains

(33)
$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{2} e^{i(x-t)} - \right.$$

$$- \frac{i\pi}{4} e^{\frac{-it^2}{4x}} e^{-(\frac{1-2V}{\gamma})^2 t} \left[2 - \operatorname{erfc}\{-i(\frac{1-2V}{\gamma}) t^{1/2}\} \right] +$$

$$\left. + e^{-i(\frac{t^2}{4x} - \frac{\pi}{4})} \left[0(t^{-1/2}) \right] \right\} \quad (V < 1/2)$$

Therefore

(34)
$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{4} e^{\frac{-it^2}{4x}} e^{-(\frac{1-2V}{\gamma})^2 t} \operatorname{erfc}\{-i(\frac{1-2V}{\gamma}) t^{1/2}\} + \right.$$

$$\left. + e^{-i(\frac{t^2}{4x} - \frac{\pi}{4})} \left[0(t^{-1/2}) \right] \right\} \quad (V < 1/2)$$

However, (34) is identical to (29) - the case $V > 1/2$. Therefore, the restriction $V \gtrless 1/2$ is unnecessary and can be removed and either expression completely describes the

waves. The final result then is uniformly valid about the wavefront whose position is defined by $V = x/t = 1/2$. Some simplification of $\eta(x\ t)$ is possible and there are many equivalent expressions for it. A particularly compact expression is given below.

$$(35) \qquad \eta(x,t) \sim I_m \left\{ \frac{i\pi}{4}\, e^{i(x-t)}\, \mathrm{erfc}[\sqrt{x}(1-t/2x)e^{\frac{i\pi}{4}}] + \right.$$

$$\left. + \; e^{-i\left(\frac{t^2}{4x} - \frac{\pi}{4}\right)} \left[\frac{\sqrt{\pi x/t}\{1-4x/t-4x^2/t^2\}t^{-1/2}}{2(1+2x/t)\ (1+4x^2/t^2)} + O(t^{-3/2}) \right] \right\}$$

For the cases $V \ll 1/2$, $V \gg 1/2$, and $V = 1/2$, expressions for the waveheight $\eta(x,t)$ can be developed in an alternative manner. The effect of the simple pole at $\sigma = 1$ is either negligible or an easily evaluated residue contribution in these cases. These alternative expansions for $\eta(x,t)$ have intrinsic merit and in addition provide a check on the validity of the uniformly valid expression that has been derived. They are available in Appendix E.

EXPERIMENTAL PROCEDURE

Experiments were conducted in the Ship Model Towing Tank located in Building 48, Hydrodynamics Laboratory. The purpose of the experiments was to provide a comparison between the waves generated by a wavemaker and those predicted by the derived theoretical expression.

Control of the oscillatory motions of the wavemaker paddle was accomplished by utilizing the sinusoid voltage output of a function generator as the input to the paddle motion control mechanism - a hydraulic device. A voltage output proportional to the deflection of the wavemaker paddle was fed back and compared to the input voltage providing close synchronization of the paddle motion to the desired motion. The amplitude of horizontal displacement of the paddle at the water surface was measured. Both frequency of paddle oscillations and maximum amplitude of displacement were controllable parameters throughout the experiment. It was not found necessary to vary the maximum amplitude of displacement from its initial value of 1 inch, however. Paddle frequency was varied from 0.8 cycles per second (cps) to 2.0 cps. Above a frequency of 1.2 cps nonlinear effects were encountered and severe vibrations of the wavemaker machinery noted. Transverse wave patterns were also induced indicating that significant three-

dimensional features were present in the flow. As a result, all experimental data for runs above 1.2 cps was not evaluated. Experiments were not conducted for frequencies below 0.8 cps so as to avoid the effects of finite towing tank depth.

The height of the waves generated by the wavemaker was measured with waveprobes. These instruments are variable resistance devices whose electrical resistance changes with depth of immersion into the water. The voltage output of the waveprobes was connected to the input of a strip chart recorder (Sanborn, series 7700). Thus a continuous record of the waves at a particular location from the wavemaker paddle versus time is obtained. Calibration of the waveprobes was accomplished by immersing the probes a measured distance into the water and adjusting the deflection of the strip chart recorder scribe to a convenient value. The deflection of the wavemaker boundary was also converted to a voltage signal and connected to the chart recorder input so that the motions of the wavemaker were continuously monitored.

RESULTS

To facilitate comparison of the waves predicted by equation (35) and those measured experimentally, all dimensional quantities are normalized to a consistent base and the results illustrated graphically. The transformation equations relating dimensional to nondimensional quantities are those used in the theoretical analysis in the derivation of equation (35) and are as follows:

$$x = \frac{w^2 \bar{x}}{g}$$ where w = frequency of oscillation of wavemaker

$$t = w\,\bar{t}$$ g = acceleration due to gravity

$$\eta = \frac{\pi}{2}\frac{\bar{\eta}}{S}$$ S = amplitude of oscillation of wavemaker at the free surface

All bar quantities are dimensional

Predicted values of the wave height are readily computed manually using tables to determine values of the error function complement for complex arguments. Equation (35) is well suited, however, for computer evaluation of $\eta(x,t)$. Computer programs for this purpose are presented in Appendix G. In addition, a function subprogram for the error function complement for complex arguments is listed separately.

Computer facilities were also used to plot the theoretical and experimental values of wave height using the

Stromberg-Carlson 4020 (SC-4020) cathode ray tube plotting device. Graphs of experimental results are computer faired with points corresponding to normalized, measured values of wave height encircled. Comparison of faired to experimental results show that the faired results differ very little from measured values. Reproductions of the wave records obtained during the experiments are available in Appendix F.

Figures 1 through 5 compare theoretical and experimental results for five values of x. Wave height $\eta(x,t)$ is plotted versus time t for fixed values of distance x. These figures therefore show the timewise distribution of wave height as seen by an observer located at a fixed distance from the wavemaker. Figure 6 is generated from equation (35) for $x = 65$ and illustrates well all the prominant features of the timewise distribution of waves. Figure 7 shows the spacewise distribution of wave height for various of time t. These figures are arranged sequentially and show the wave front propagating out from the wavemaker with increasing time.

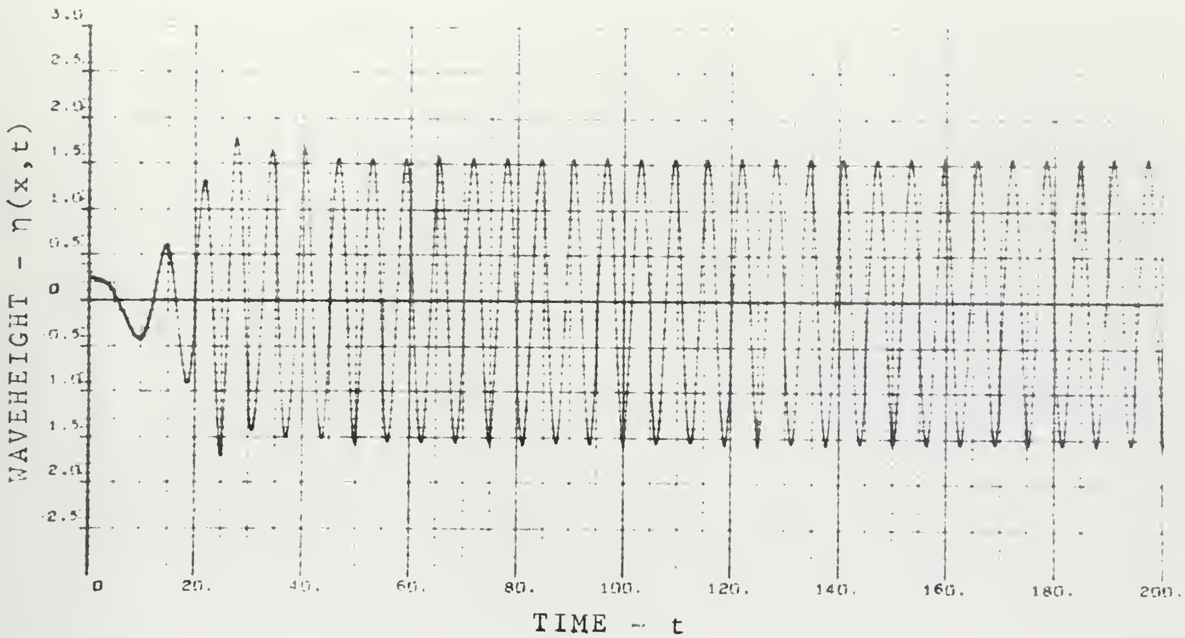


Fig. 1-A. Wave Height vs. Time for x = 8.7687. Based on Equation (35).

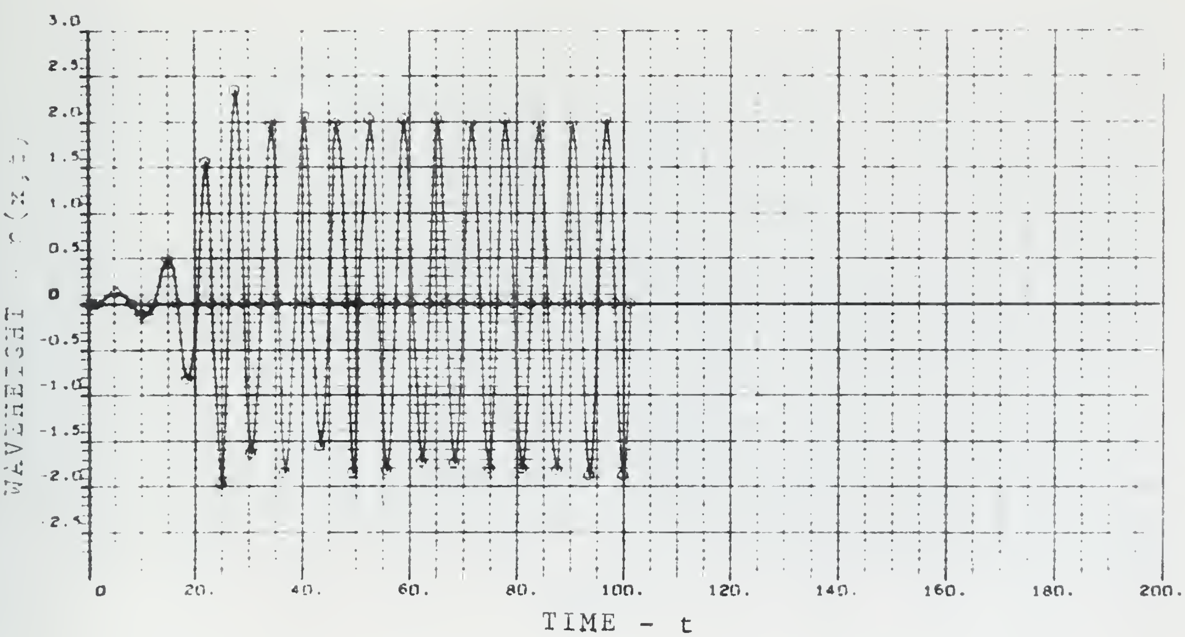


Fig. 1-B. Wave Height vs. Time for x = 8.7687. Based on Measured Values ($\bar{x}$ = 11.6667 ft, f = 0.7826 cps).

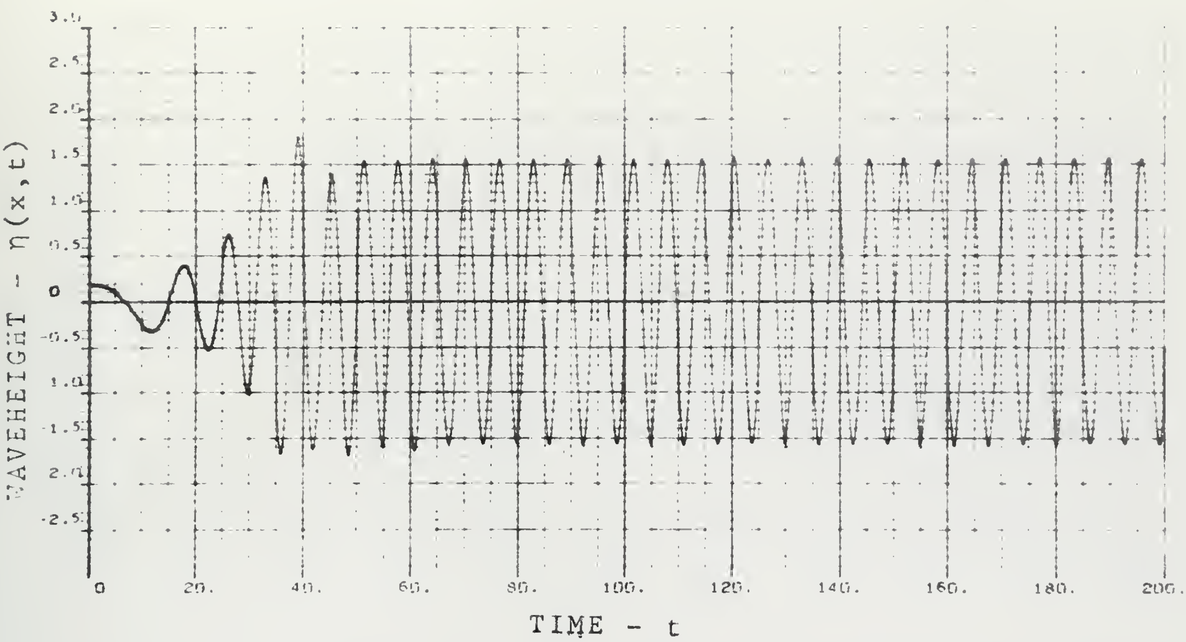


Fig. 2-A. Wave Height vs. Time for x = 13.6493. Based on Equation (35).

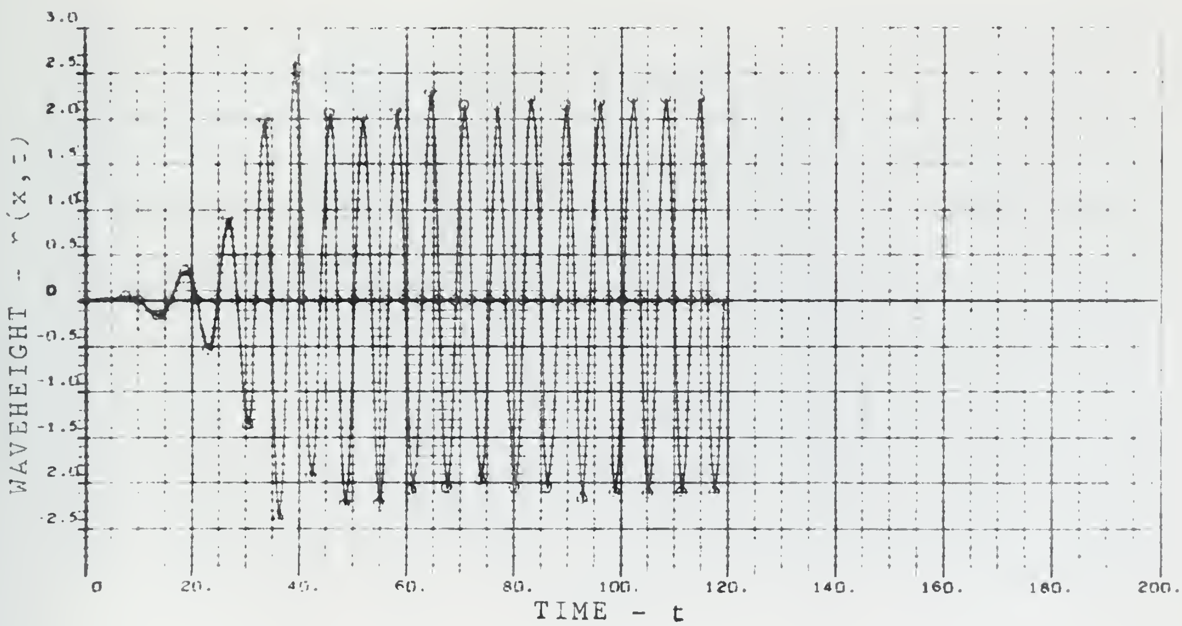


Fig. 2-B. Wave Height vs. Time for x = 13.6493. Based on measured values ($\bar{x}$ = 11.6667 ft, f = 0.9764 cps).

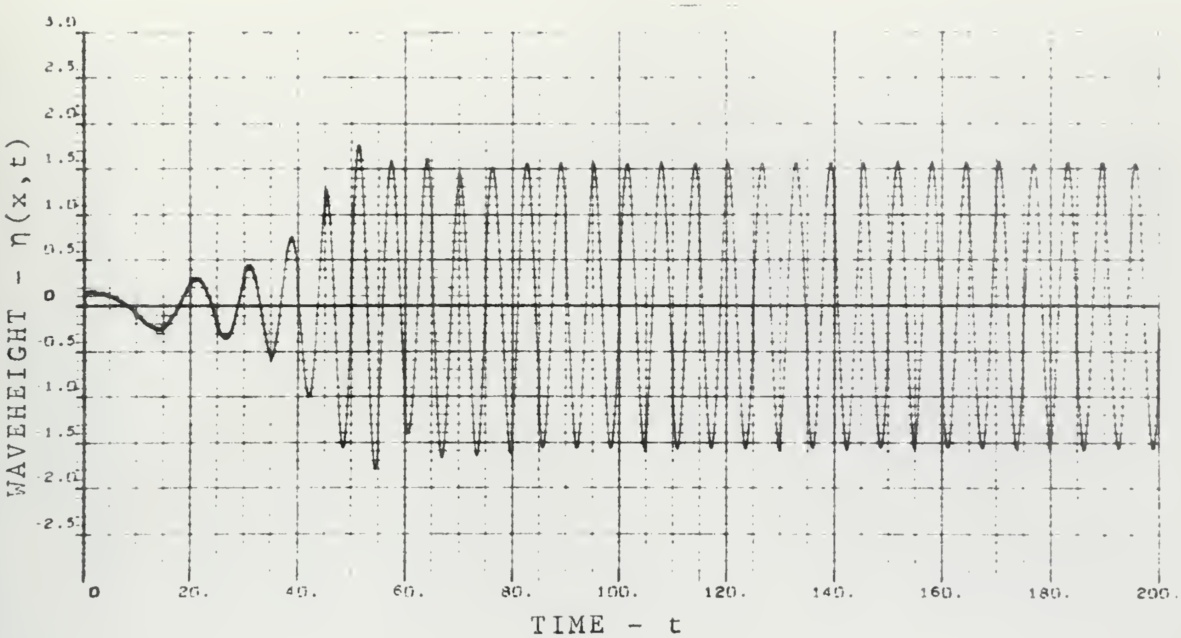


Fig. 3-A. Wave Height vs. Time for x = 19.7632. Based on Equation (35).

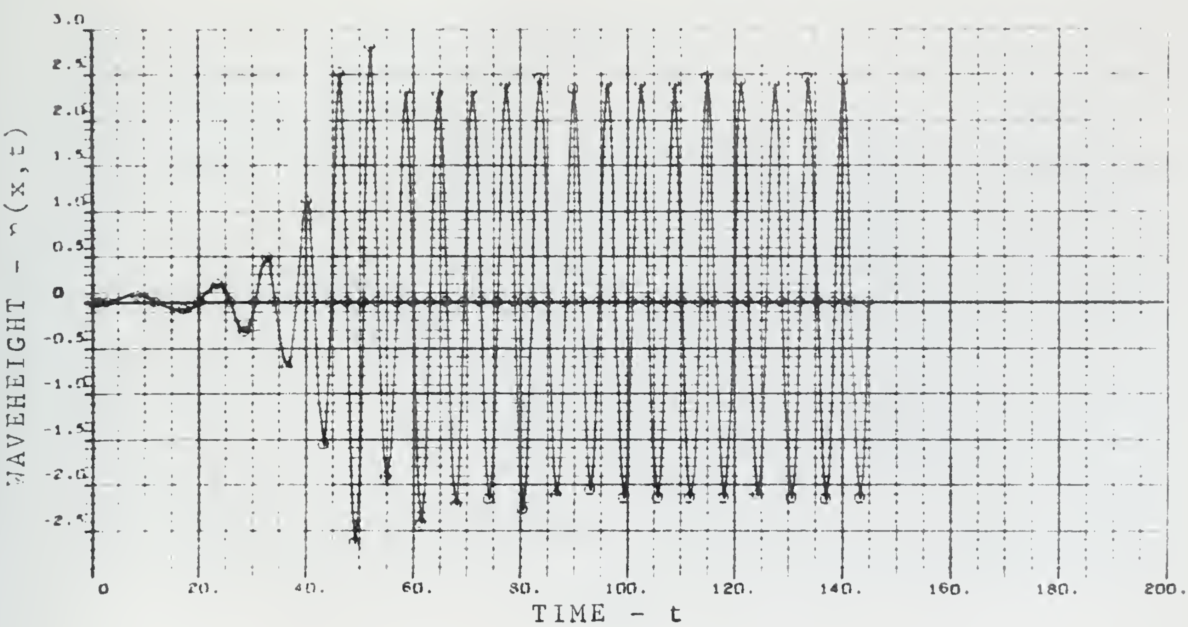


Fig. 3-B. Wave Height vs. Time for x = 19.7632. Based on measured values ($\overline{x}$ = 11.6667 ft, f = 1.1749 cps).

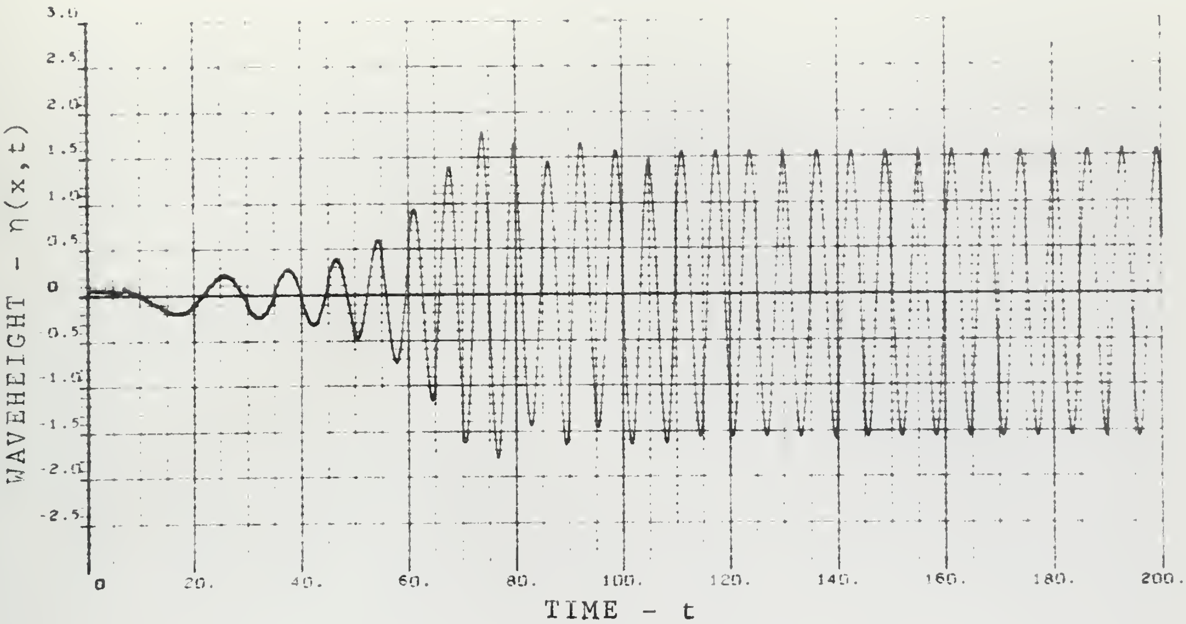


Fig. 4-A. Wave Height vs. Time for x = 29.4239. Based on Equation (35).

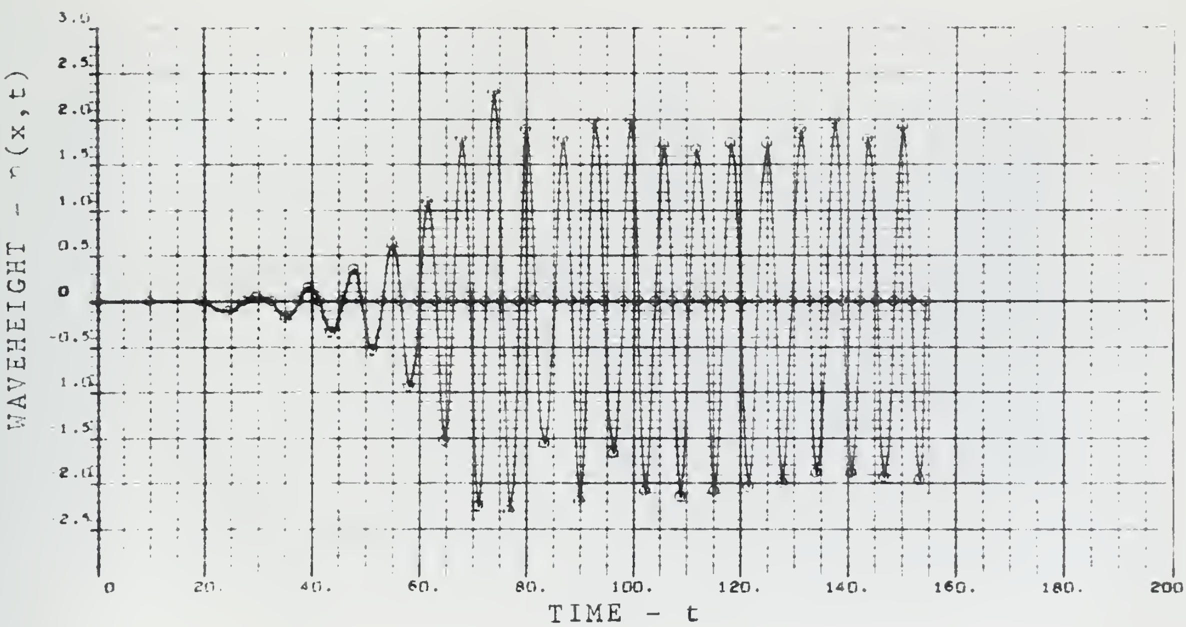


Fig. 4-B. Wave Height vs. Time for x = 29.4239. Based on measured values ($\bar{x}$ = 38.1667 ft, f = 0.7926 cps).

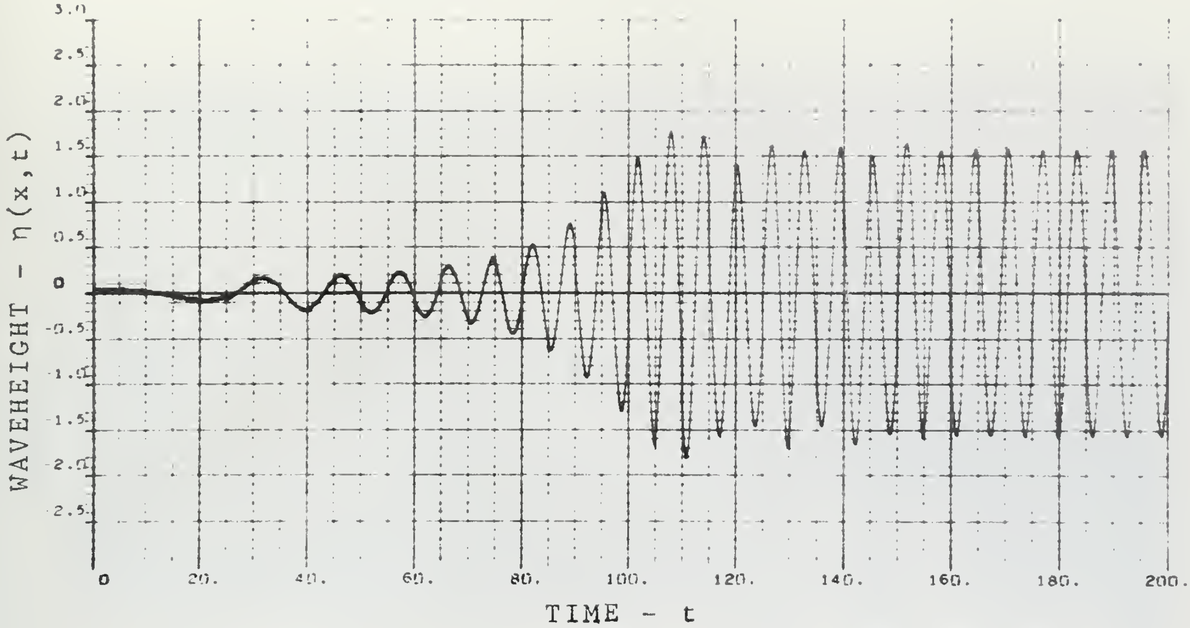


Fig. 5-A. Wave Height vs. Time for x = 44.8634. Based on Equation (35).

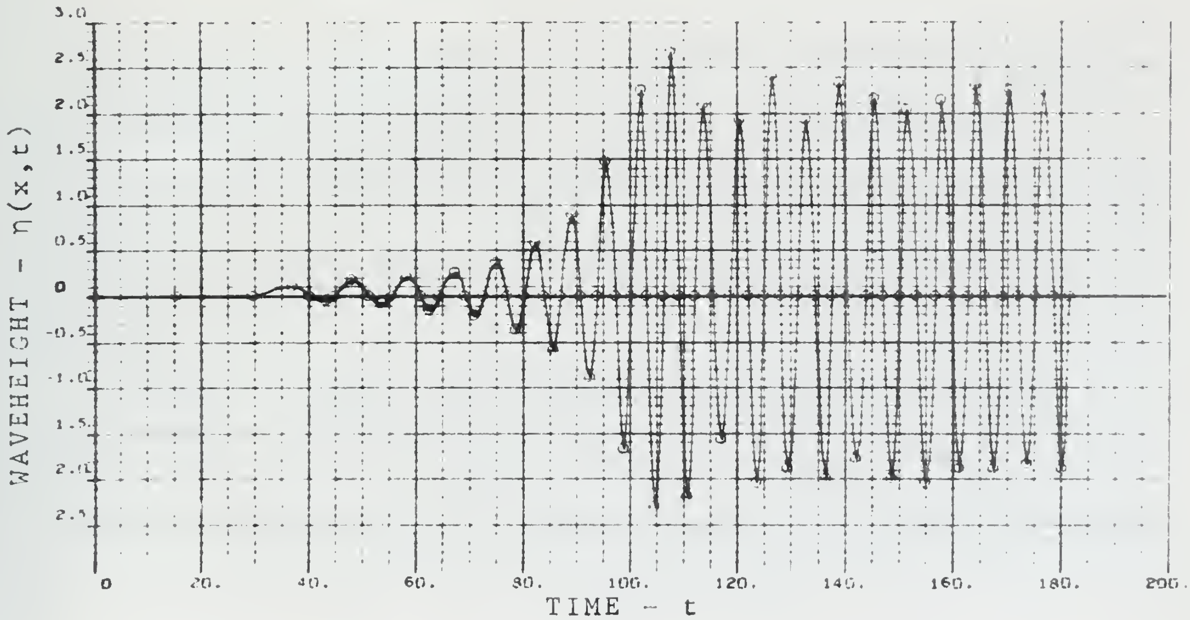


Fig. 5-B. Wave Height vs. Time for x = 44.8634. Based on measured values ($\overline{x}$ = 38.1667 ft, f = 0.9787 cps).

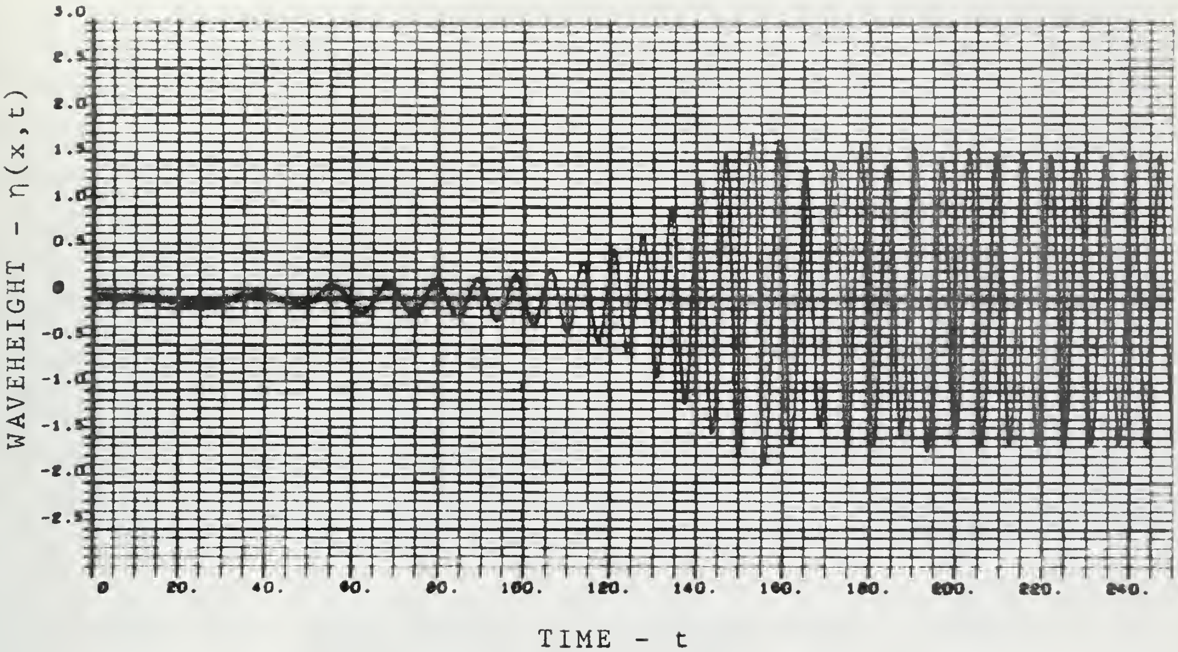


Fig. 6. Wave Height vs. Time for x = 65. Based on Equation (35).

Figure 6 illustrates well all the prominant features of the timewise distribution of waves. The wave height increases monotonically to a maximum value and then enters an oscillatory phase until the steady state wave amplitude is reached.

Figure 7 shows the spacewise distribution of wave height for various values of time t. These figures are arranged sequentially and show the wave front propagating out into calm water with increasing time.

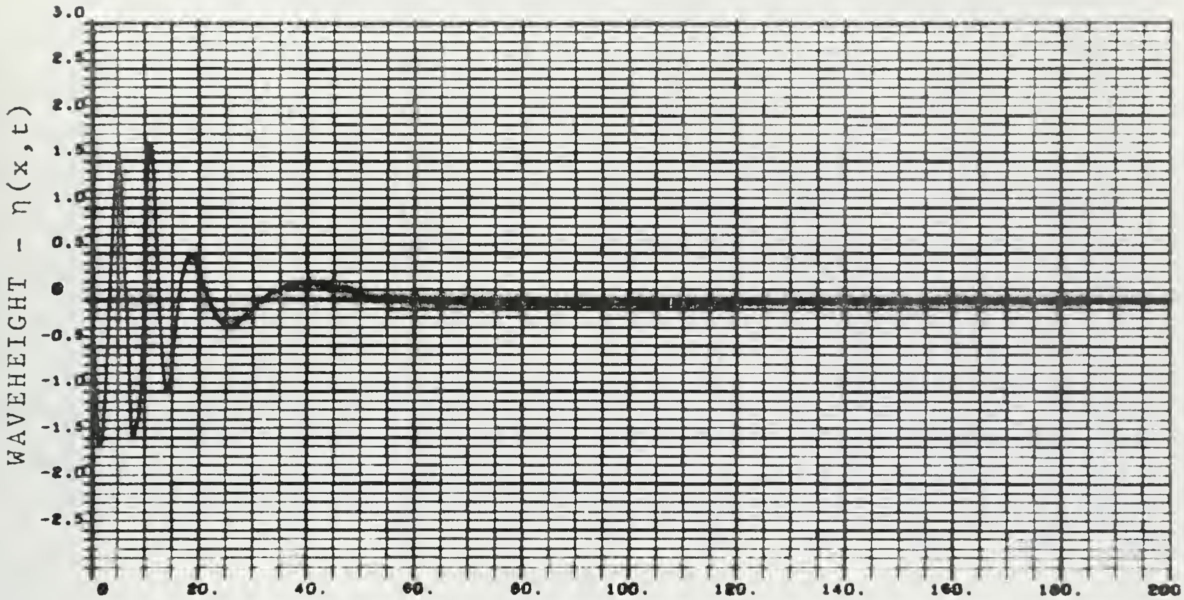


Fig. 7-A. Wave Height vs. Distance for t = 30.0. Based on Equation (35).

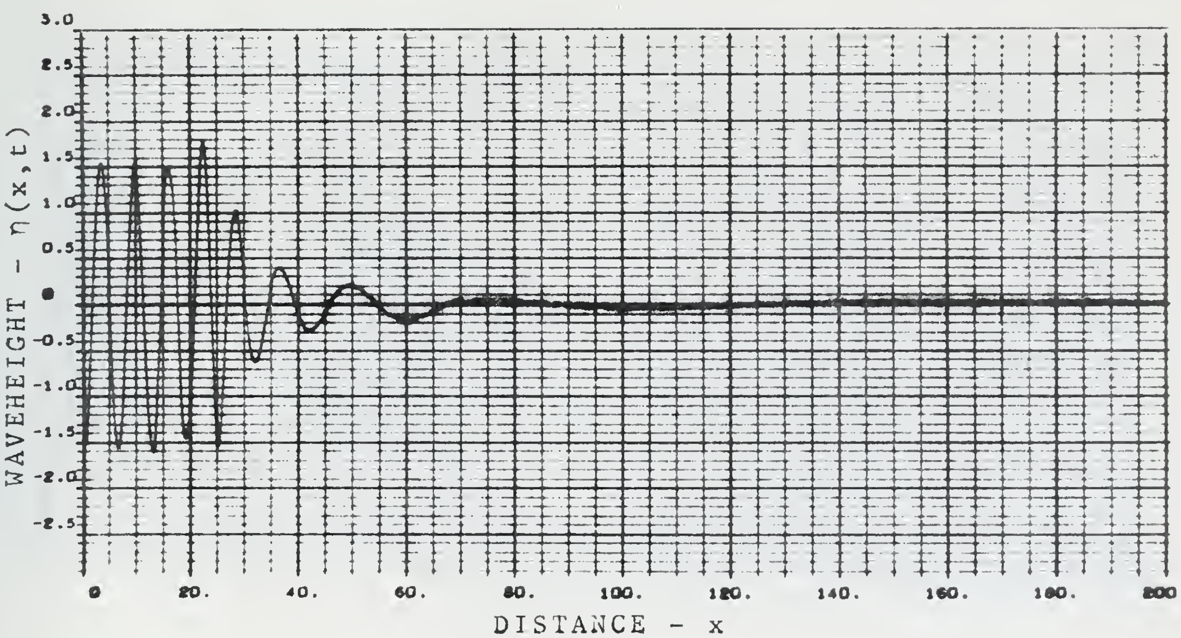


Fig. 7-B. Wave Height vs. Distance for t = 60.0. Based on Equation (35).

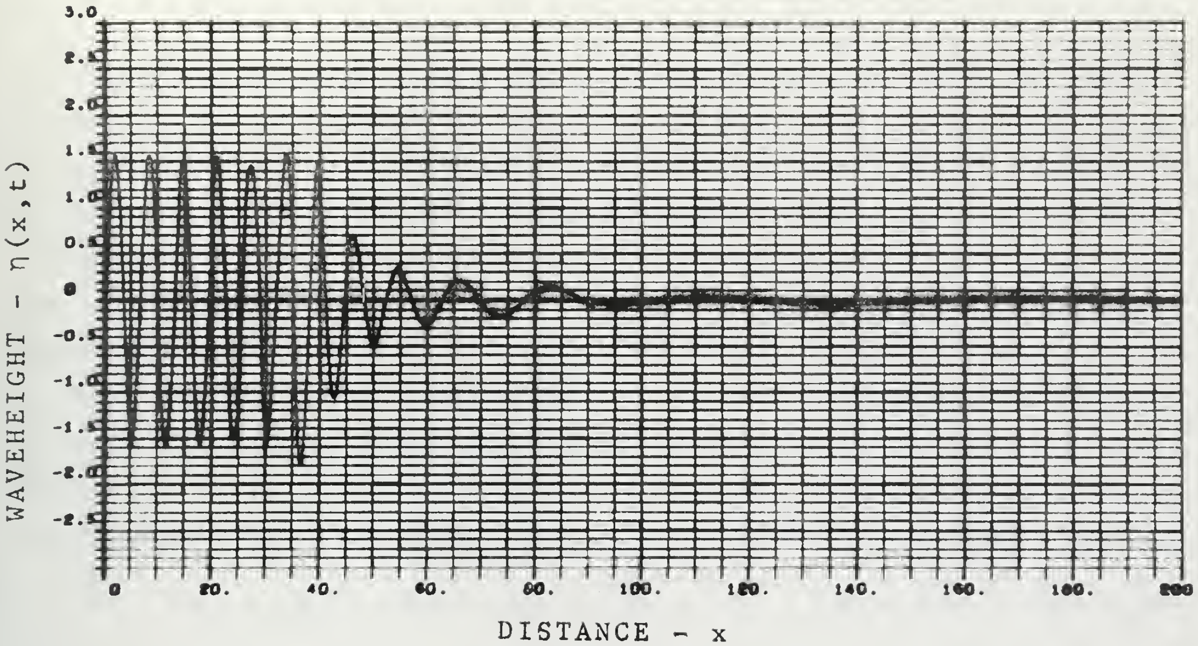


Fig. 7-C. Wave Height vs. Distance for $t = 90.0$ Based on Equation (35).

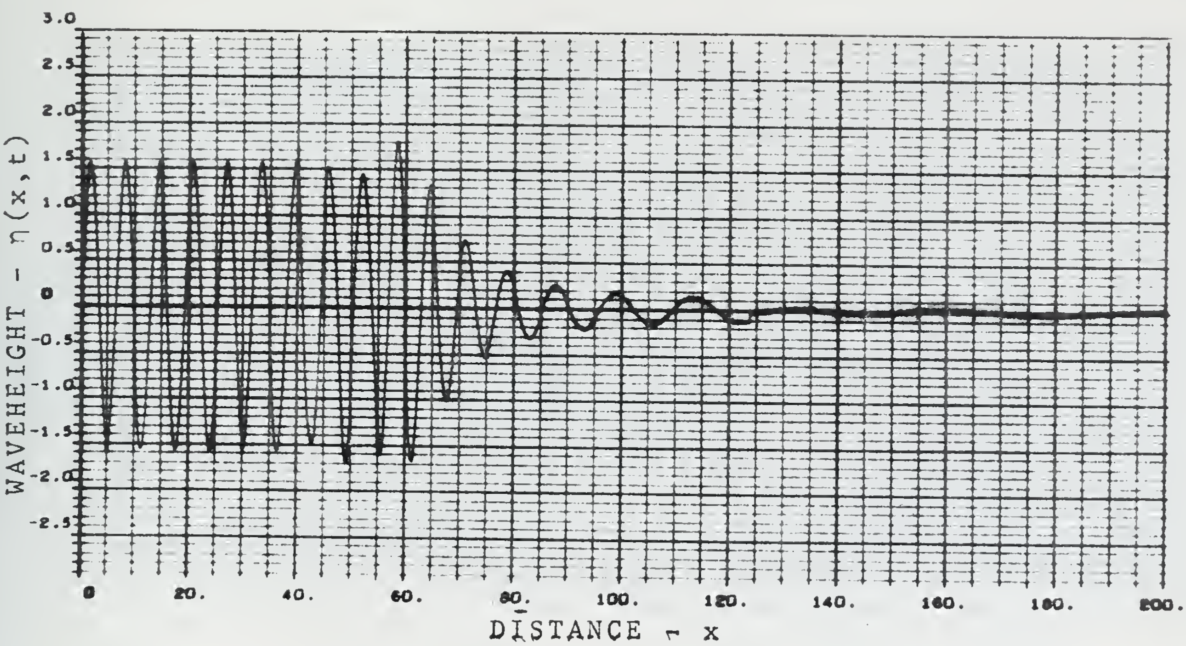


Fig. 7-D. Wave Height vs. Distance for $t = 140.0$. Based on Equation (35).

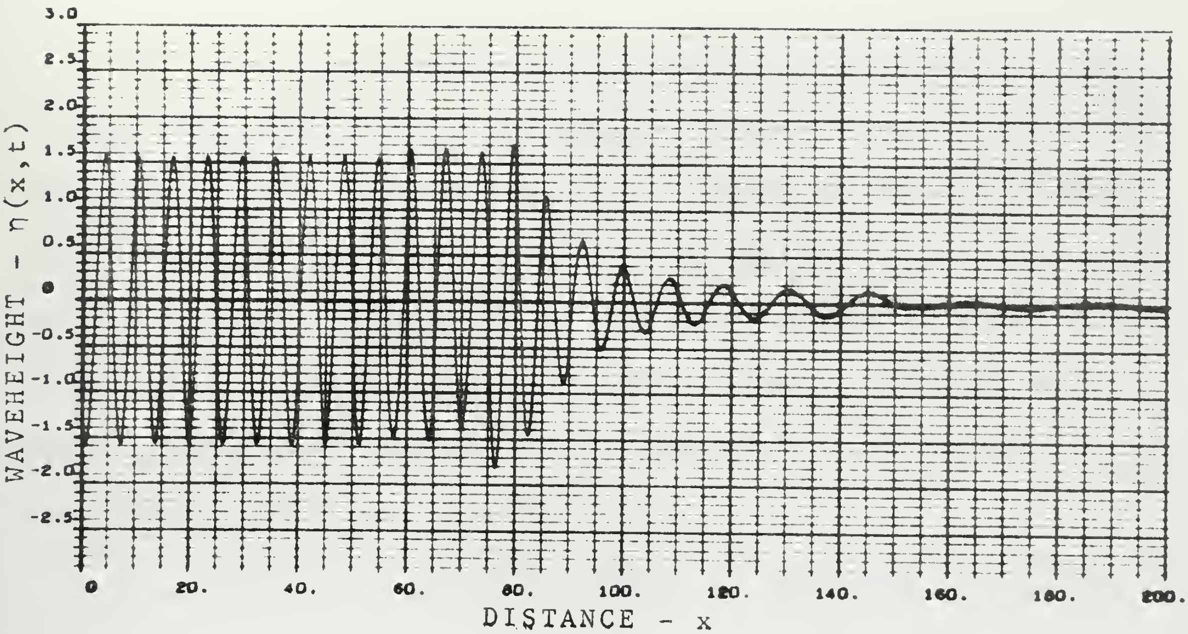


Fig. 7-E. Wave Height vs. Distance for t = 180.0.
Based on Equation (35).

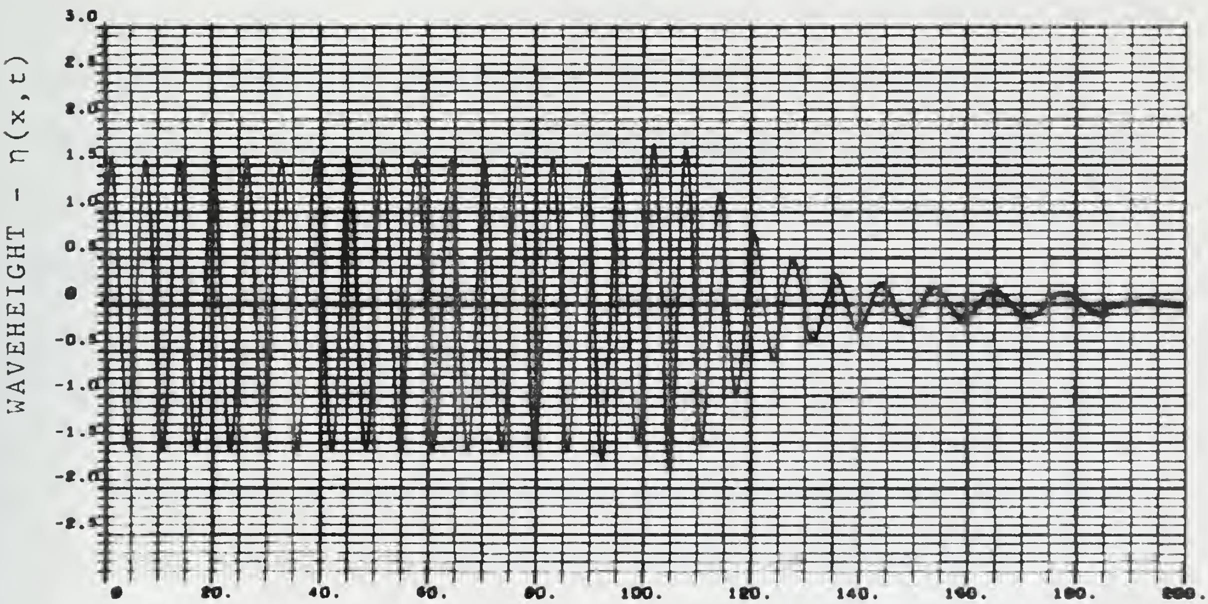


Fig. 7-F. Wave Height vs. Distance for t = 240.0.
Based on Equation (35).

DISCUSSION

The waves predicted by the equation (35) for the wave height as a function of distance and time compare well with measured values. To an observer located a fixed distance from the wavemaker, the wave height increases monotonically with time to a maximum value and then enters an oscillatory phase during which the steady state wave amplitude is approached. Both measured and predicted wave records show this characteristic behavior.

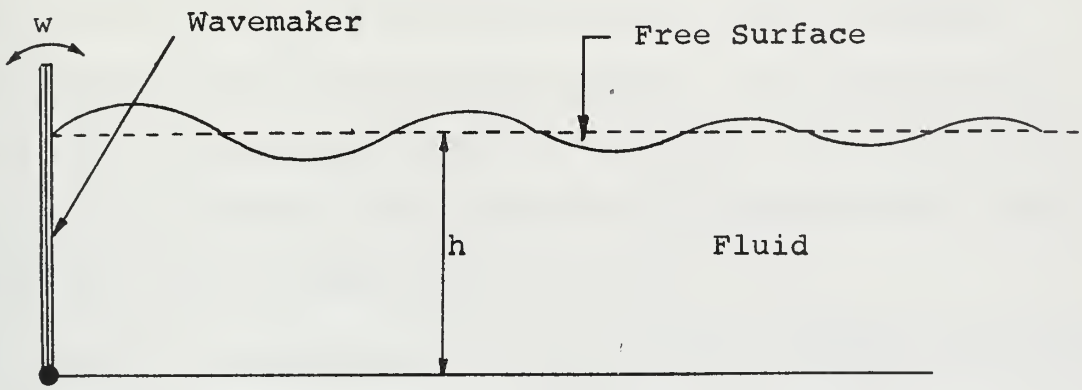
Measured and predicted values of the wave height correspond very closely in phase, except for very small values of t when equation (35) is no longer valid. However, measured values of wave amplitude are consistently larger than predicted values. This difference can be attributed to the boundary condition imposed on the moving surface of the wavemaker in the theoretical development. For convenience of solution, this boundary condition was assumed to be the following:

$$s = S\, e^{w^2 y/g} \sin(wt)\, H(t) \qquad (-\infty < y \leq 0) \tag{1}$$

However, a more realistic approximation to the flow near the towing tank wavemaker paddle is given by

$$s = \begin{cases} S\dfrac{(y + h)}{h} \sin(wt)\, H(t) & (y \geq -h) \\ 0 & (y \leq -h) \end{cases} \tag{36}$$

This is not the exact boundary condition but is a reasonable approximation to it. It assumes that displacements of the wavemaker from the vertical are small with respect to the depth of the wavemaker paddle (approximately equal to the depth h of the towing tank wave generator) and is the linear approximation to the actual displacement.



In addition, the theoretical expression for $\eta(x,t)$ is for the case of infinite depth water. The waves generated by the towing tank wavemaker have been chosen so that finite depth effects have a negligible effect on the wave features. Therefore, the boundary condition given by (36) is a simplified model of the physical situation shown above.

A theoretical development based on the more realistic approximation to the boundary condition poses no additional complication to the problem but does significantly add to the algebraic tedium and perhaps obscures evaluation of the

results somewhat. Of primary concern is the determination as to just how the use of boundary condition (1) above in lieu of (36) has influenced the problem. Physical considerations suggest that assuming an exponential decrease in fluid velocity with depth instead of the more realistic linear decay should result in wave amplitude predictions of smaller magnitude than measured values. As less energy is imparted to the fluid near the moving surface (average amplitude of oscillation being smaller), wave amplitudes must be correspondingly smaller as energy considerations are ultimately related to the square of wave amplitude. This is substantiated by experimental results.

Partial verification of the physical argument presented above can be readily obtained in spite of the algebraic complexities of the more realistic boundary condition. The steady state wave amplitude can be easily obtained by letting t approach infinity and evaluating the remaining terms. For the case at hand this amounts to determining the residue contributions to the integral representation for the wave height.

The solution to the unsteady wavemaker problem (infinite depth water) can be placed in a more general form where the functional dependence of the wavemaker paddle displacement on depth is left unspecified (subject to the

restrictions previously mentioned on this function). One formulation of it is the following:

(37)

$$\eta(x,t) = -\frac{ws}{\pi}\int_0^{\infty}\cos(ux)\left[\int_0^{-\infty} e^{uy}F(y)dy\right]\left[\frac{\sin(wt)-\sin(\sqrt{gu}\ t)}{w-\sqrt{gu}} + \right.$$

$$\left. + \frac{\sin(wt)+\sin(\sqrt{gu}\ t)}{w+\sqrt{gu}}\right] du$$

For $F(y) = e^{\frac{w^2}{g}y}$,

$$\int_0^{-\infty} e^{uy}F(y) = -\frac{1}{u+w^2/g}$$

and equation (2) is rederived.

$$\text{For } F(y) = \begin{cases} \dfrac{y+h}{h} & y \geq -h \\ 0 & y \leq -h \end{cases}$$

$$\int_0^{-\infty} e^{uy}F(y)dy = -\frac{1}{hu^2}(uh + e^{-uh} - 1)$$

Therefore, the wave height $\eta(x,t)$ is given by

(38) $\eta(x,t) =$

$$= \frac{2ws}{\pi}\int_0^{\infty}\frac{(uh+e^{-uh}-1)}{hu^2}\cos(ux)\left[\frac{\sqrt{gu}\ \sin(\sqrt{gu}\ t)-w\ \sin(wt)}{gu-w^2}\right]du$$

Introducing the change in variable $u = \sigma^2/g$ and using non-dimensional variables $x = w^2/g\ x$, $t = wt$, $\sigma = \sigma/w$, $h = w^2/g\ h$, and $\eta = \pi\eta/2s$ one obtains:

(39)
$$\eta(x,t) = I_m \left\{ \int_0^\infty \frac{\{\sigma^2 h + e^{-\sigma^2 h} - 1\}}{h\sigma^4(\sigma+1)(\sigma-1)} \left\{ \sigma e^{i(\sigma^2 x + \sigma t)} - \right.\right.$$

$$\left.\left. - \sigma e^{i(\sigma^2 x - \sigma t)} - e^{i(\sigma^2 x + t)} + e^{i(\sigma^2 x - t)} \right\} d\sigma \right\}$$

Following methods similar to those used in deriving (35), as $t \to \infty$

(40)
$$\eta(x,t) \sim I_m \{ i\pi \operatorname{Res}[1] \}$$

$$\sim \frac{(h + e^{-h} - 1)\ \pi}{h} \cos(x-t)$$

where h = depth of towing tank.

For the measured depth of 43.5 inches (3.62 feet) and wavemaker paddle frequency of 0.9787 cps, $h = 4.25$.

(41) $$\eta(x,t) \sim \frac{(4.25 + e^{-4.25} - 1}{4.25} \times \pi \cos(x-t) \simeq 2.4 \cos(x-t)$$

Equation (41) compares favorably with measured steady state values of wave amplitude corresponding to a paddle

frequency of 0.9787 cps. These experimentally obtained values of wave height are given by:

(42) $$\eta(x,t) \approx 2.35 \cos(x-t)$$

The ratio of the steady state wave height associated with boundary condition (36) to that associated with boundary condition (1) is 2.4/1.57 = 1.53. If this ratio is assumed to hold in a very loose sense throughout the transient zone, then the predicted values of wave height are very close to measured values.

For large values of t or x, equation (35) for $\eta(x,t)$ must approach the historical solutions of Lamb and of Cauchy and Poisson. This property can be shown by expanding the erfc for large positive and negative values of its argument. Letting t approach ∞ one obtains:

(43) $$\eta(x,t) \sim \pi/2 \cos(x-t)$$

These are the outgoing waves at infinity that Lamb derived. Letting x approach ∞ one obtains as the first order term:

(44) $$\eta(x,t) \sim \frac{sg^{1/2}}{2w\pi^{1/2}} x^{-1/2} \cos(gt^2/4x - \pi/4)$$

Equation (44) exhibits the general features of the Cauchy/Poisson solution.

CONCLUSIONS

In his analysis of the transient gravity wave response to an oscillating pressure, Miles[5] described many of the qualitative features of the generated waves as were observed in the towing tank and predicted by the derived expression for $\eta(x,t)$, Equation (35). The wave amplitudes associated with a concentrated oscillating pressure also show the monotonic increase to a maximum value followed by an oscillatory phase to the steady state amplitude.

In addition, Miles defines a rise-time T as the time for the timewise envelope to rise from 0.1 of its steady state value to its first maximum, and a width X of the transition zone for the spacewise envelope as the distance between the point where the wave height reaches 0.1 of its steady state value and its maximum amplitude. The expressions for T and X are as follows:

$$T \simeq 9\ (x/g)^{1/2} \tag{45}$$

$$X \simeq 3\ g\ w^{-3/2} t^{1/2} \tag{46}$$

It is of interest to determine if the theory developed for the transient wave response to a wavemaker supports these observations or modifies them in any meaningful way.

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[5] Miles, *Op. Cit.*, pp. 149-150.

Equation (35) gives the transient wave response to a wavemaker and can be rewritten as follows:

$$(47)\quad \eta(x,t) \sim I_m \left\{ \frac{i\pi}{4} e^{i(x-t)} \operatorname{erfc}[\sqrt{x}(1-t/2x)e^{i\pi/4}] + 0(t^{-1/2}) \right\}$$

Terms of order $0(t^{-1/2})$ can be neglected for sufficiently large t (or x). It is convenient to treat the remainder of (47) as an outgoing plane wave with amplitude and phase given functions of x and t. With this view in mind, (47) can be rewritten as follows:

$$(48)\quad \eta(x,t) \sim A \cos[(x-t) + \psi]$$

where

$$(49)\quad A = \pi/4 \; | \operatorname{erfc}[\sqrt{x}(1-t/2x)e^{i\pi/4}] \; |$$

$$(50)\quad \psi = \operatorname{Arg} \{ \operatorname{erfc}[\sqrt{x}(1-t/2x)e^{i\pi/4}] \}$$

Expressions corresponding to (45) and (46) can be developed for the transient wave response to a wavemaker by investigating the behavior of A and using Miles' definitions of X and T.

The behavior of A as a function of $U = \sqrt{x}(1-t/2x)$ is shown below.

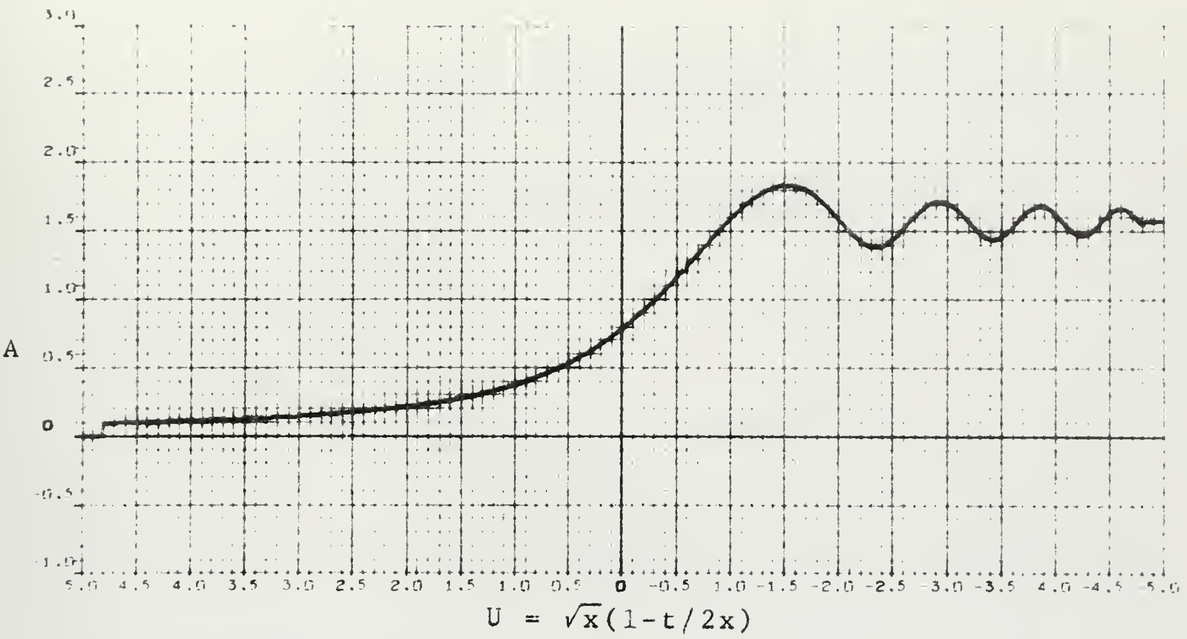

Fig. 8. Timewise Wave Envelope

A is equal to 0.1 of its steady state value ($A = 0.15708 = 0.1\times\pi/2$) when $U = +2.795$. A reaches its maximum value when $U = -1.5255$. With these values, the rise-time T and the width of the transient zone X are readily determined. In terms of x and t, the maximum and 0.1 steady state amplitudes occur when:

(51) $\sqrt{x}\,(1-t/2x) = -1.5255$ for $A = A_{max}$

(52) $\sqrt{x}\,(1-t/2x) = 2.795$ for $A = 0.1\times\pi/2 = A_{0.1ss}$

For $x = x_o$:

$$t_m = 2x_o + (2)(1.5255)x_o^{1/2}$$

$$t_{0.1ss} = 2x_o - (2)(2.795)x_o^{1/2}$$

Therefore

$$T = t_m - t_{0.1ss} = 8.641x_o^{1/2}$$

(53) $$T \approx 9x_o^{1/2}$$

For $t = t_o$:

$$\sqrt{x}_m(1 - t_o/2x_m) = -1.5255 \quad \text{for } A = A_{max}$$

$$\sqrt{x}_{0.1ss}(1 - t_o/2x_{0.1ss}) = 2.795 \quad \text{for } A = A_{0.1ss}$$

These relations for x_m reduce to

$$x_m^2 - \{t_o + (-1.5255)^2\}x_m + t_o^2/4 = 0$$

$$x_{0.1ss}^2 - \{t_o + (2.795)^2\}x_{0.1ss} + t_o^2/4 = 0$$

x_m and $x_{0.1ss}$ are then given by

$$x_m = \frac{\{t_o + (-1.5255)^2\} - 1.5255\sqrt{2t_o + (-1.5255)^2}}{2}$$

$$x_{0.1ss} = \frac{\{t_o + (2.795)^2\} + 2.795\sqrt{2t_o + (2.795)^2}}{2}$$

Therefore,

$$X = x_{0.1ss} - x_m$$

$$= \frac{5.495 + 2.795\sqrt{2t_o+7.82} + 1.5255\sqrt{2t_o+2.325}}{2}$$

For sufficiently large t_o:

(54) $$X = 3.06\, t_o^{1/2} \simeq 3\, t_o^{1/2}$$

Equations (53) and (54) can be placed in the same form as (45) and (46) by introducing dimensional quantities, thus obtaining:

(55) $$T \simeq 9\left(\frac{x_o}{g}\right)^{1/2}$$

(56) $$X \simeq 3gw^{-3/2}t_o^{1/2}$$

The theory developed for the transient wave response to a wavemaker thus supports the observations of Miles with respect to the asymptotic behavior of the wave envelope A(x,t). The conclusions regarding the rise-time T and the width of the transient zone X apply without qualification to the transient waves generated by both oscillating pressure sources and moving surfaces.

RECOMMENDATIONS

Several subjects for future analysis are recommended which are related to the problem considered by this study. They include the following:

1. Develop a uniformly valid expression for the transient wave response to a wavemaker similar to equation (35), but using the more realistic approximation to the flow near the wavemaker paddle given by equation (36) in lieu of (1). Compare the waves given by this expression to the experimental results obtained in this study.

2. Consider the problem of the transient gravity wave response to an oscillating pressure treated by Miles but using the method developed in this study - first obtaining an integral representation for the wave height using Green's theorem and an appropriate Green's function.

3. Consider the transient wave response to a wavemaker in a fluid with finite depth.

4. Consider the transient wave response to an oscillating pressure in a fluid with finite depth.

APPENDIX A

THE TIME-DEPENDENT GREEN'S FUNCTION

The solution of the general problem of the infinitesimal, irrotational, unsteady motion of a fluid with a free surface cannot usually be given in explicit form; however, it can be shown that the problem can be reduced to that of finding an appropriate Green's Function. The essential tool used in solving the problem is the time-dependent Green's Function, $G(x,y,z;\xi,\eta,\zeta;t,\tau)$, which is of intrinsic interest itself. It represents the velocity potential which is the solution to the following problem: At time $t = \tau$ a disturbance in the nature of a source is originated at a point (x,y,z) which coincides with (ξ,η,ζ). It is assumed that the pressure on the free surface is always zero and that no immersed surfaces are present. G satisfies homogeneous boundary conditions on all boundaries and appropriate conditions at infinity. After G is constructed, the velocity potential for the given problem can be represented in terms of initial conditions and boundary conditions. If no submerged bodies are present an explicit solution is obtained; otherwise the problem is reduced to solving an integral equation.[6] The two-dimensional time-dependent Green's Function will now be developed for infinite depth fluid and fluid of finite depth h.

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[6] Finkelstein, A. B., "The Initial Value Problem for Transient Water Waves", *Comm. on Pure and Applied Math.*, Vol. X, pp. 511-522.

$G(x,y;\xi,\eta,;t,\tau)$ - Infinite Depth Fluid.

G must satisfy the following governing equations:

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-\xi)\delta(y-\eta) \qquad \begin{array}{l} -\infty<y<0 \\ -\infty<x<+\infty \end{array}$$

(here δ represents the Dirac Delta Function)

$G_{tt} + gG_y = 0$ on $y = 0$, the free surface boundary condition

$G \to 0$ as $y \to -\infty$

$G = G_t = 0$ for $t = \tau$, $y = 0$, the initial conditions

Introducing the Fourier transform f of G

$$f = \int_{-\infty}^{\infty} Ge^{-iux}dx$$

one obtains

$$G = \frac{1}{2\pi}\int_{-\infty}^{\infty} fe^{iux}du$$

Then substituting for G its Fourier transform representation, the governing equations become

1) $e^{iu\xi}(f_{yy} - u^2 f) = \delta(y - \eta)$

2) $f_{tt} + gf_y = 0$ on $y = 0$

3) $f \to 0$ as $y \to -\infty$

4) $f = f_t = 0$ for $t = \tau$, $y = 0$

The general solution to the above problem can be expressed in the following alternative ways:

$$e^{iu\xi}f = ae^{uy} + be^{-uy} = a'\cosh(uy) + b'\sinh(uy)$$

$$= a''\cosh[u(y+h)] + b''\sinh[u(y+h)]$$

for $y < \eta$, solve equation (1) subject to equation (3) only.

$$e^{iu\xi}f = a\, e^{|u|y}$$

for $y > \eta$, solve equation (1) subject to equation (2) only.

$$e^{iu\xi}f = c \cosh(uy) - \frac{c_{tt}}{gu} \sinh(uy)$$

Require f to be continuous at $y = \eta$ and that the derivative of f have a discontinuity of magnitude -1 at $y = \eta$, that is

$$e^{iu\xi}f_y\Big]_{y=\eta+} - e^{iu\xi}f_y\Big]_{y=\eta-} = -1$$

From the continuity requirement above

$$e^{iu\xi}f = \begin{cases} e^{|u|y}\left[c \cosh(u\eta) - \dfrac{c_{tt}}{gu} \sinh(u\eta)\right] & (y < \eta) \\ e^{|u|\eta}\left[c \cosh(uy) - \dfrac{c_{tt}}{ug} \sinh(uy)\right] & (y > \eta) \end{cases}$$

From the requirement on the derivative of f

$$c_{tt} + guc = g$$

$$\text{subject to} \quad c = c_t = 0 \quad \text{for } t = \tau$$

The solution for c is

$$c = \frac{1}{u}\left(1 - \cos[\sqrt{gu}(t-\tau)]\right)$$

Therefore, the expression for f reduces to

$$e^{iu\xi}f = \begin{cases} e^{|u|y}\left[-\dfrac{\sinh(u\eta)}{u} + \dfrac{e^{|u|\eta}}{u}\{1 - \cos[\sqrt{gu}(t-\tau)]\}\right] & (y<\eta) \\ e^{|u|\eta}\left[-\dfrac{\sinh(uy)}{u} + \dfrac{e^{|u|y}}{u}\{1 - \cos[\sqrt{gu}(t-\tau)]\}\right] & (y>\eta) \end{cases}$$

Inverting to obtain $G(x,y;\xi,\eta;t,\tau)$

$$G = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{e^{uy}}{u}\sinh(u\eta)\cos[u(x-\xi)]du & (y < \eta) \\ & + \\ -\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{e^{u\eta}}{u}\sinh(uy)\cos[u(x-\xi)]du & (y > \eta) \end{cases}$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{e^{u(y+\eta)}}{u} [1 - \cos\sqrt{gu}(t-\tau)] \cos u(x-\xi)du$$

Recalling the known integral

$$\int_0^\infty \frac{e^{-ax}-e^{-a'x}}{x} \cos(mx)dx=\log\left[\frac{\sqrt{a'^2+m^2}}{\sqrt{a^2+m^2}}\right]$$

then G may be expressed as

$$G(x,y;\xi,\eta;t,\tau) = -\frac{1}{2\pi}\log\left[\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\right] +$$

$$+ \frac{1}{\pi}\int_0^\infty e^{u(y+\eta)} \frac{[1 - \cos\sqrt{gu}(t-\tau)]}{u} \cos[u(x-\xi]du$$

<u>$G(x,y;\xi,\eta;t,\tau)$ - Fluid of Finite Depth h.</u>

G must satisfy the following governing equations:

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-\xi)\delta(y-\eta) \quad \text{for} \quad \begin{array}{l} -h < y < 0 \\ -\infty < x < +\infty \end{array}$$

$G_{tt} + gG_y = 0$ on $y = 0$, the free surface boundary condition

$G_y = 0$ on $y = -h$, the bottom boundary condition

$G = G_t = 0$ for $t = \tau$, $y = 0$, the initial conditions

Substituting for G its Fourier transform representation f, the governing equations become

$$1)\ e^{iu\xi}(f_{yy} - u^2 f) = \delta(y - \eta)$$

$$2)\ f_{tt} + g f_y = 0 \qquad \text{on } y = 0$$

$$3)\ f_y = 0 \qquad \text{on } y = -h$$

$$4)\ f = f_t = 0 \qquad \text{for } t = \tau,\ y = 0$$

For $y < \eta$, the solution to eq. 1) subject to eq. 3) is

$$e^{iu\xi} f = a \cosh[u(y+h)]$$

For $y > \eta$, the solution to eq. 2) is

$$e^{iu\xi} f = c \cosh(uy) - \frac{c_{tt}}{gu} \sinh(uy)$$

Require f to be continuous at $y = \eta$ and that the derivative of f have a discontinuity of magnitude -1, that is

$$e^{iu\xi} f_y \Big]_{y=\eta+} - e^{iu\xi} f_y \Big]_{y=\eta-} = -1$$

From the continuity requirement above

$$e^{iu\xi} f = \begin{cases} \cosh\{u(y+h)\}[c \cosh(u\eta) - \frac{c_{tt}}{gu} \sinh(u\eta)] & (y < \eta) \\ \cosh\{u(\eta+h)\}[c \cosh(uy) - \frac{c_{tt}}{gu} \sinh(uy)] & (y > \eta) \end{cases}$$

From the requirement on the derivative of f

$$c_{tt} + gu\tanh(uh)c = \frac{g}{\cosh(uh)}$$

Subject to $c = c_t = 0$ for $t = \tau$

The solution for c is

$$c = \frac{1}{u\sinh(uh)}\{1 - \cos[\sigma(t-\tau)]\}$$

where $\sigma = \sqrt{gu\tanh(uh)}$

Therefore, the expression for f reduces to

$$e^{iu\xi}f = \begin{cases} \cosh[u(y+h)]\left[-\dfrac{\sinh(u\eta)}{u\cosh(uh)} + \right. \\ \left. + \dfrac{\cosh[u(\eta+h)]\{1 - \cos[\sigma(t-\tau)]\}}{u\sinh(uh)\cosh(uh)}\right] & (y < \eta) \\ \cosh[u(\eta+h)]\left[-\dfrac{\sinh(uy)}{u\cosh(uh)} + \right. \\ \left. + \dfrac{\cosh[u(y+h)]\{1 - \cos[\sigma(t-\tau)]\}}{u\sinh(uh)\cosh(uh)}\right] & (y > \eta) \end{cases}$$

Inverting to obtain $G(x,y;\xi,\eta;t,\tau)$

$$G = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\cosh[u(y+h)]\sinh(u\eta)}{u\cosh(uh)}\cos[u(x-\xi)]du & (y < \eta) \\ -\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\cosh[u(\eta+h)]\sinh(uy)}{u\cosh(uh)}\cos[u(x-\xi)]du & (y > \eta) \end{cases} +$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{\cosh[u(\eta+h)]\cosh[u(y+h)]}{\cosh^2(uh)}\frac{[1-\cos\{\sigma(t-\tau)\}]}{u\tanh(uh)}\cos[u(x-\xi)]du$$

Recalling the useful identity

$$\cosh[u(y+h)] = e^{uy}\cosh(uh) - e^{-uh}\sinh(uy)$$

then G can be expressed in the following form

$$G = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{e^{uy}\sinh(u\eta)}{u}\cos[u(x-\xi)]du & (y < \eta) \\ -\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{e^{u\eta}\sinh(uy)}{u}\cos[u(x-\xi)]du & (y > \eta) \end{cases} +$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{e^{-uh}\sinh(uy)\sinh(u\eta)}{u\cosh(uh)}\cos[u(x-\xi)]du +$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{\cosh[u(y+h)]\cosh[u(\eta+h)]}{\cosh^2(uh)} \cdot \frac{[1-\cos\{\sigma(t-\tau)\}]}{u\tanh(uh)}\cos[u(x-\xi)]du$$

where $\sigma = \sqrt{gu\tanh(uh)}$

Using the known integral expression for the log given previously, G can finally be expressed as

$$G(x,y;\xi,\eta;t,\tau) = -\frac{1}{2\pi}\log\left[\frac{\sqrt{(x-\xi)^2 + (y-\eta)^2}}{\sqrt{(x-\xi)^2 + (y+\eta)^2}}\right]$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{e^{-uh}\sinh(u\eta)\sinh(uy)}{u\cosh(uh)}\cos[u(x-\xi)]du +$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{\cosh[u(\eta+h)]\cosh[u(y+h)]}{\cosh^2(uh)} \frac{[1-\cos\{\sigma(t-\tau)\}]}{u\tanh(uh)}\cos[u(x-\xi)]du$$

where $\sigma = \sqrt{gu\tanh(uh)}$

APPENDIX B

THE UNSTEADY WAVEMAKER SOLUTION

Consider a semi-infinite body of incompressible, inviscid, fluid having a free surface and infinite depth. Let the limiting boundary execute infinitesimal displacements from an initial vertical plane position given by $s = SF(y,t)$ where S has dimensions of length and therefore $F(y,t)$ is dimensionless. Initially $F(y,0+) = 0$. It is assumed that $F(y,t)$ is sufficiently smooth and vanishes sufficiently rapidly as y approaches $-\infty$. The problem is described mathematically in terms of a velocity potential as follows:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad y \leq 0,\ x \geq 0$$

subject to $\phi_{tt} + g\phi_y = 0$ on $y = 0$, the free surface boundary condition

$\phi \to 0$ as $y \to -\infty$

$\phi_x = SF(y,t)$ on $x = 0$

$\phi_y = \phi_t = 0$ for $t = 0+$; $y = 0$, the initial conditions

The relevant time-dependent Green's Function associated with the above problem is easily obtained by the method of images and is given by:

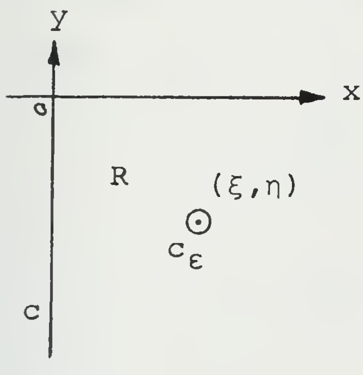
$$G(x,y;\xi,\eta;t,\tau) = -\frac{1}{2\pi}\log\left[\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}\sqrt{(x+\xi)^2+(y-\eta)^2}}{\sqrt{(x-\xi)^2+(y+\eta)^2}\sqrt{(x+\xi)^2+(y+\eta)^2}}\right] +$$

$$+ \frac{1}{\pi}\int_0^\infty e^{u(y+\eta)}\,\frac{[1-\cos\{\sqrt{gu}(t-\tau)\}]}{u}\,[\cos\{u(x-\xi)\} +$$

$$+ \cos\{u(x+\xi)\}]du$$

Note: The only singularity in the region occupied by the fluid is at $(x,y) = (\xi,\eta)$. All other singular points are outside the boundary enclosing the fluid.

Applying Green's Theorem of vector analysis (in 2-dimensional form) to the region occupied by the fluid exclusive of the singular point (ξ,η) one obtains:



Green's Theorem

$$\iint_R (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)\,dA =$$

$$\oint_c (\phi_1\frac{\partial\phi_2}{\partial n} - \phi_2\frac{\partial\phi_1}{\partial n})\,ds$$

Let $\phi_1 = G(x,y;\xi,\eta;t,\tau)$

$\phi_2 = \phi_t(x,y;t)$

Then substituting into the above expression for Green's Theorem,

$$\iint_{R'} (G\nabla^2\phi_t - \phi_t\nabla^2 G)\,dA = \oint_c (G\phi_{tn} - \phi_t G_n)\,ds + \oint_{c_\varepsilon} (G\phi_{tn} - \phi_t G_n)\,ds$$

where $R' = R$ less the singular point (ξ,η); c_ε is a circle of vanishingly small radius enclosing (ξ,η).

However, $\nabla^2\phi = 0$ implies $\nabla^2\phi_t = 0$ and $\nabla^2 G = 0$ except at (ξ,η). Therefore

$$\oint_c (G\phi_{tn} - \phi_t G_n)ds = \oint_{c_\varepsilon} (G\phi_{tn} - \phi_t G_n)ds$$

On c_ε, $ds = r d\theta$, $\partial/\partial n = -\frac{\partial}{\partial r}$ where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$

With this substitution the above expression reduces to

$$\oint_c (G\phi_{tn} - \phi_t G_n)\, ds = \lim_{r\to 0} \left[\int_0^{2\pi} -(G\phi_{tr} - \phi G_r)\, r d\theta \right]$$

ϕ_t is regular throughout R; therefore, the only contributions to the above limit must come from G and G_r (assuming the limit of the integral is equal to the integral of the limit) and are given by

$$(1)\quad \lim_{r\to 0} [rG] \sim \lim_{r\to 0} \left[r\left(\frac{-\log r}{2\pi}\right) \right] \to 0$$

$$(2)\quad \lim_{r\to 0} [rG_r] \sim \lim_{r\to 0} \left[r \left(-\frac{1}{2\pi r}\right) \right] \to -\frac{1}{2\pi}$$

Therefore

$$\oint_c (G\phi_{tn} - \phi_t G_n)ds = \frac{-\phi_t(\xi,\eta;t)}{2\pi} \int_0^{2\pi} d\theta = -\phi_t(\xi,\eta;t)$$

Thus

$$\phi_t(\xi,\eta;t) = \oint (\phi_t G_n - G\phi_{tn})ds$$

Integrating the above expression from 0 to τ

$$\int_0^\tau \phi_t(\xi,\eta;t)\,dt = \int_0^\tau \left[\oint_C (\phi_t G_n - G\phi_{tn})\,ds \right] dt$$

Note that from the behavior of G and G_n at ∞ and from the prescribed boundary conditions it is easily shown that $\oint_C$ reduces to

$$\oint_C = \cancelto{0}{\int_{y=-\infty}} + \cancelto{0}{\int_{x=+\infty}} + \int_{y=0} + \int_{x=0}$$

On x=0, $\partial/\partial n = -\partial/\partial x$. On y=0, $\partial/\partial n = +\partial/\partial y$. Therefore

$$\int_0^\tau \phi_t(\xi,\eta;\tau) = \phi(\xi,\eta;\tau) - \phi(\xi,\eta;0) =$$

$$= \int_0^\tau \left[-\underset{(y=0)}{\int_0^\infty} (\phi_t G_y - G\phi_{ty})\,dx - \underset{(x=0)}{\int_0^{-\infty}} (\phi_t G_x - G\phi_{tx})\,dy \right] dt =$$

$$= \underset{(y=0)}{\int_0^\infty} \left[\int_0^\tau (G\phi_{ty} - \phi_t G_y)\,dt \right] dx + \underset{(x=0)}{\int_0^{-\infty}} \left[\int_0^\tau (G\phi_{tx} - \phi_t G_x)\,dt \right] dy =$$

$$= \underset{(y=0)}{\int_0^\infty} \left[\left(G\phi_y + \frac{\phi_t G_t}{g}\right) \Big|_0^\tau - \int_0^\tau G_t \left(\phi_y + \frac{\phi_{tt}}{g}\right) dt \right] dx +$$

$$+ \underset{(x=0)}{\int_0^{-\infty}} \left[(G\phi_x - \phi G_x) \Big|_0^\tau - \int_0^\tau (G_t \phi_x - \phi G_{tx})\,dt \right] dy$$

For the problem at hand

$$G_x(0,y;\xi,\eta;t,\tau) = G_{tx}(0,y;\xi,\eta;t,\tau) = 0$$

$$\phi_x(0,y;0) = 0$$

$$\phi_y + \frac{\phi_{tt}}{g} = 0 \quad \text{on } y = 0$$

$$G(x,0;\xi,\eta;\tau,\tau) = G_t(x,0;\xi,\eta;\tau,\tau) = 0$$

$$\phi_y(x,0;0) = \phi_t(x,0;0) = 0$$

$\phi(x,y;0) = 0$ (as the fluid is assumed initially at rest)

Therefore

$$\phi(\xi,\eta;\tau) = \int_{\substack{0\\(x=0)}}^{-\infty} [G(0,y;\xi,\eta;\tau,\tau)\phi_x(0,y;\tau) - \int_0^{\tau} G_t\phi_x dt]dy$$

Let

$$s = S\, e^{u_o y} \sin(wt)\, H(t)$$

$$v = wS\, e^{u_o y} \cos(wt)\, H(t) = \phi_x(0,y;t)$$

$$a = -w^2 S\, e^{u_o y} \sin(wt)\, H(t) + $$
$$+ wS\, e^{u_o y} \cos(wt)\delta(t) = \phi_{tx}(0,y;t)$$

where

$u_o = w^2/g$

$H(t)$ is the Heaviside step function

$\delta(t)$ is the Dirac Delta function

Substituting these expressions into $\phi(\xi,\eta;\tau)$ above, it is easily shown that

$$\phi(\xi,\eta;\tau) = \frac{-wS\cos(w\tau)H(\tau)}{2\pi}\int_0^{-\infty} e^{u_o y}\log\left\{\frac{\xi^2+(y-\eta)^2}{\xi^2+(y+\eta)^2}\right\}dy +$$

$$+ \frac{-2wS}{\pi}\int_0^{\infty}\frac{e^{u\eta}\cos(u\xi)}{u}\left[\int_0^{\infty} e^{(u+u_o)y}\,dy\right] \times$$

$$\times \left[\int_0^{\tau}\sqrt{gu}\,\sin[\sqrt{gu}(t-\tau)]\cos(wt)H(t)dt\right] du$$

Of more direct interest, however, are the waves in lieu of the velocity potential. The expression for the waveheight is obtained from the linearized free surface boundary condition given by

$$\eta(\xi,\tau) = -\frac{1}{g}\phi_\tau(\xi,0;\tau)$$

Thus

$$\eta(\xi,\tau) = \frac{2wS}{\pi}\int_0^{\infty}\frac{\cos(u\xi)}{u+u_o}\left[\int_0^{\tau}\cos\{\sqrt{gu}(t-\tau)\}\cos(wt)H(t)dt\right] du$$

or

$$\eta(\xi,\tau) = \frac{wS}{\pi}\int_0^{\infty}\frac{\cos(u\xi)}{u+u_o}\left[\frac{\sin(w\tau)-\sin(\sqrt{gu}\tau)}{w-\sqrt{gu}} + \frac{\sin(w\tau)+\sin(\sqrt{gu}\tau)}{w+\sqrt{gu}}\right]du$$

where $u_o = w^2/g$

APPENDIX C

ASYMPTOTIC BEHAVIOR OF CERTAIN INTEGRALS

1. The asymptotic behavior of the integral given below is desired.

$$\int_{C_1} \frac{\sigma^2}{\sigma^4-1} e^{i(\sigma^2 x-\sigma t)} d\sigma$$

The contour C_1 has been chosen so that it corresponds to the steepest descent path from the origin to infinity. Introduce the change in variable $i\tau = \sigma^2(x/t) - \sigma$. With this substitution, the complex variable τ is real along this contour. The resulting integral expression is the following:

$$\int_0^\infty (\sigma^2/\sigma^4-1)\ d\sigma/d\tau\ e^{-t\tau}\ d\tau$$

The asymptotic behavior of this integral can be ascertained by expanding the integrand about $\tau = 0$ and integrating the resulting expression term by term. However, the integrand is regular about this point and therefore the Taylor series expansion is applicable. The series expansion is given below and is readily obtained through repeated differentiation of the integrand.

$$\frac{\sigma^2}{\sigma^4-1}\frac{d\sigma}{d\tau} = f(\tau) = f(0) + f'(0)\tau + f''(0)\tau^2 + \ldots$$

$$= 0 + 0 - i\tau^2 + \ldots$$

The first nonzero term of the Taylor expansion is of order $0(\tau^2)$. The dominant term in the asymptotic expansion of $\int_{C_1}$ is then proportional to:

$$\int_{C_1} \frac{\sigma^2}{\sigma^4-1} e^{i(\sigma^2 x - \sigma t)} d\sigma = -i \int_0^\infty \tau^2 e^{-t\tau} d\tau + \ldots$$

Introducing the change in variable $t\tau = y$, the above expression can be expressed as follows:

$$\int_{C_1} \frac{\sigma^2}{\sigma^4-1} e^{i(\sigma^2 x - \sigma t)} d\sigma = -it^{-3} \int_0^\infty y^2 e^{-y} dy + \ldots$$

$$= -it^{-3}\Gamma(3) + \ldots$$

The dominant term in the asymptotic expansion of this integral is therefore of order $0(t^{-3})$.

2. Consider the following integrals:

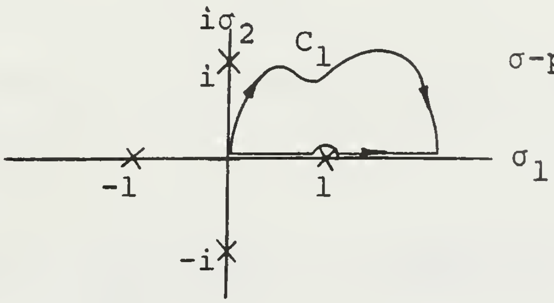
$$I_1 = \int_0^\infty f(\sigma)\, e^{i(\sigma^2 x + \sigma t)}\, d\sigma$$

$$I_3 = \int_0^\infty \frac{f(\sigma)}{\sigma}\, e^{i(\sigma^2 x + t)}\, d\sigma$$

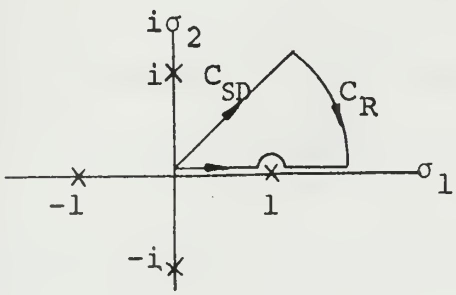
$$I_4 = \int_0^\infty \frac{f(\sigma)}{\sigma}\, e^{i(\sigma^2 x - t)}\, d\sigma$$

where $f(\sigma) = \sigma^2/\sigma^4-1$

The asymptotic behavior of these integrals can be obtained by deforming the path of integration in the complex σ plane along a contour with the general features of C_1 below.



A particularly convenient equivalent contour for the evaluation of I_1, I_3, and I_4 consists of the steepest descents path from the origin to infinity and then along the arc $R\ e^{i\theta}$ to the real axis. Such a contour is shown below.



where C_{SD} is the steepest descents path

$C_R = Re^{i\theta}$, $R \to \infty$

For integrals I_3 and I_4, the steepest descent path is simply the ray $\sigma = r\ e^{i\pi/4}$. For integral I_1, the steepest descent path is not as easily described but no further complication is added to the analysis.

As the integrand is in all cases regular at $\sigma = 0$, it may be expanded in a Taylor series expansion about $\sigma = 0$ and the resulting expression integrated term by term. By inspection, the first two terms of the Taylor series are equal to zero for I_1. Only the first term is zero for

integrals I_3 and I_4. The contribution to the integral is then of the order $0(t^{-3})$ for integral I_1 and of order $0(t^{-2})$ for integrals I_3 and I_4.

The asymptotic behavior of integrals I_1, I_2, and I_4 can be described as below:

$$I_1 \sim i\pi \operatorname{Res}(1) + \underset{(R\to\infty)}{\int_{C_R}^{0}} + 0(t^{-3}) = (\frac{i\pi}{4})e^{i(x+t)} + 0(t^{-3})$$

$$I_3 \sim i\pi \operatorname{Res}(1) + \underset{(R\to\infty)}{\int_{C_R}^{0}} + 0(t^{-2}) = (\frac{i\pi}{4})e^{i(x+t)} + 0(t^{-2})$$

$$I_4 \sim i\pi \operatorname{Res}(1) + \underset{(R\to\infty)}{\int_{C_R}^{0}} + 0(t^{-2}) = (\frac{i\pi}{4})e^{i(x-t)} + 0(t^{-2})$$

APPENDIX D

LAURENT EXPANSIONS

To obtain an asymptotic expansion for $\int_{C_2}$, it is convenient to obtain the Laurent expansion of

$$\frac{\sigma^2}{\sigma^4-1}\,\frac{d\sigma}{d\tau}$$

about $\tau = 0$ such that the effect of the pole at $\sigma = 1$ has been subtracted off with the remainder of the expression regular at $\sigma = 1$. Algebraic difficulties are minimized if σ^2/σ^4-1 is first expanded by the method of partial fractions. It is easily shown that

$$\frac{\sigma^2}{\sigma^4-1} = \frac{1}{4}\left[\frac{i}{\sigma+i} + \frac{-i}{\sigma-i} + \frac{-1}{\sigma+1} + \frac{1}{\sigma-1}\right]$$

1. Consider first, the expansion of $[\sigma_+^2/\sigma_+^4-1]\,\dfrac{d\sigma_+}{d\tau}$

$$\frac{\sigma_+^2}{\sigma_+^4-1}\,\frac{d\sigma_+}{d\tau} = \left[\frac{i}{(1+2iv)+\gamma\tau^{1/2}} + \frac{-i}{(1-2iv)+\gamma\tau^{1/2}} + \right.$$

$$\left. + \frac{-1}{(1+2v)+\gamma\tau^{1/2}} + \frac{+1}{(1-2v)+\gamma\tau^{1/2}}\right]\frac{\gamma}{8\tau^{1/2}}$$

This expression is easily placed in the form

$$\frac{\sigma_+^2}{\sigma_+^4-1}\,\frac{d\sigma_+}{d\tau} = \frac{-\gamma/8}{1-2v}\left[\tau^{1/2} + \left(\frac{1-2v}{\gamma}\right)\right]^{-1} +$$

$$+ \frac{\tau^{-1/2}}{8}\left[\frac{\gamma}{1-2v} + \frac{i}{\frac{1+2iv}{\gamma} + \tau^{1/2}} +\right.$$

$$\left. + \frac{-i}{\frac{1-2iv}{\gamma} + \tau^{1/2}} + \frac{-1}{\frac{1+2v}{\gamma} + \tau^{1/2}}\right] =$$

$$= \frac{-\gamma/8}{1-2v}\left[\tau^{1/2} + \left(\frac{1-2v}{\gamma}\right)\right]^{-1} + \frac{\gamma}{8}\left[\frac{\tau^{-1/2}}{1-2v} -\right.$$

$$- \frac{\{1-4v - 4v^2\}\tau^{-1/2}}{(1+2v)(1+4v^2)} + \frac{\gamma\{(1+4v^2)^2 - 8v(1+2v)^2\}}{(1+2v)^2(1+4v^2)^2} +$$

$$\left. + \frac{-\gamma^2\{(1+4v^2)^3 - 4v(1+2v)^3(3-4v^2)\}\tau^{1/2}}{(1+2v)^3\,(1+4v^2)^3} + \ldots\right]$$

2. The expansion of $[\sigma_-^2/\sigma_-^4-1]\frac{d\sigma_-}{d\tau}$ is obtained in a similar manner.

$$\frac{\sigma_-^2}{\sigma_-^4-1}\,\frac{d\sigma_-}{d\tau} = \left[\frac{i}{(1+2iv) - \gamma\tau^{1/2}} + \frac{-i}{(1-2iv) - \gamma\tau^{1/2}} +\right.$$

$$\left. + \frac{-1}{(1+2v) - \gamma\tau^{1/2}} + \frac{1}{(1-2v) - \gamma\tau^{1/2}}\right]\frac{-\gamma}{8\tau^{1/2}}$$

By inspection, the expansion of this function is identical to the one previously obtained except for sign changes on alternate terms. The resulting expression is readily

placed in the following form:

$$\frac{\sigma_-^2}{\sigma_-^4-1}\ \frac{d\sigma_-}{d\tau} = \frac{\gamma/8}{1-2v}\left[\tau^{1/2} - \left(\frac{1-2v}{\gamma}\right)\right]^{-1} +$$

$$+\ \frac{\gamma}{8}\left[\frac{-\tau^{-1/2}}{1-2v} + \frac{\{1-4v-4v^2\}\tau^{-1/2}}{(1+2v)\ (1+4v^2)} + \frac{\gamma\{(1+4v^2)^2 - 8v(1+2v)^2\}}{(1+2v)^2\ \ (1+4v^2)^2} +\right.$$

$$\left. + \frac{\gamma^2\{(1+4v^2)^3 - 4v(1+2v)^3(3-4v^2)\}\tau^{1/2}}{(1+2v)^3\ (1+4v^2)^3} + \ldots\right]$$

3. For values of $v = x/t << 1/2$ or $v = x/t >> 1/2$ there is no particular advantage gained in subtracting off the effect of the pole at $\sigma = 1$. As before, algebraic difficulties are minimized by expressing σ^2/σ^4-1 as a sum of linear terms.

$$\frac{\sigma^2}{\sigma^4-1}\ \frac{d\sigma}{d\tau} = \left[\frac{i}{\sigma+i} + \frac{-i}{\sigma-i} + \frac{-1}{\sigma+1} + \frac{1}{\sigma-1}\right]\frac{1}{4}\ \frac{d\sigma}{d\tau}$$

Using the familiar expansion

$$\frac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \ldots$$

the following expressions may be obtained:

$$\frac{\sigma_+^2}{\sigma_+^4-1}\ \frac{d\sigma_+}{d\tau} = 2V^{3/2}\ e^{i\pi/4}\left[\frac{\tau^{-1/2}}{1-16V^4} - \frac{4V^{1/2}i^{1/2}(1+16V^4)}{(1-16V^4)^2} +\right.$$

$$\left. + \frac{4Vi\{3+12(4V^2)^2+(4V^2)^4\}\tau^{1/2}}{(1-16V^4)^3} + O(\tau)\right]$$

$$\frac{\sigma_-^2}{\sigma_-^4-1}\,\frac{d\sigma_-}{d\tau} = -2V^{3/2}\,e^{i\pi/4}\left[\frac{\tau^{1/2}}{1-16V^4} + \frac{4V^{1/2}i^{1/2}(1+16V^4)}{(1-16V^4)^2} + \right.$$

$$\left. + \frac{4Vi\{3+12(4V^2)^2+(4V^2)^4\}\tau^{1/2}}{(1-16V^4)^3} + 0(\tau)\right]$$

4. For v = 1/2, the following Laurent expansions apply:

$$\frac{\sigma_+^2}{\sigma_+^4-1}\,\frac{d\sigma_+}{d\tau} = +\gamma^2\left[\frac{\gamma^{-2}\tau^{-1}}{8} + \frac{1}{16}\gamma^{-1}\tau^{-1/2} - \frac{3}{32} + \right.$$

$$\left. + \frac{3}{64}\gamma\tau^{1/2} - \frac{\gamma^2}{128}\tau - \frac{9}{256}\gamma^3\tau^{3/2} + \ldots\right]$$

$$\frac{\sigma_-^2}{\sigma_-^4-1}\,\frac{d\sigma_-}{d\tau} = -\gamma^2\left[-\frac{\gamma^{-2}\tau^{-1}}{8} + \frac{1}{16}\gamma^{-1}\tau^{-1/2} + \frac{3}{32} + \right.$$

$$\left. + \frac{3}{64}\gamma\tau^{1/2} + \frac{\gamma^2}{128}\tau - \frac{9}{256}\gamma^3\tau^{3/2} + \ldots\right]$$

APPENDIX E

ALTERNATIVE EXPANSIONS FOR $\eta(x,t)$

For the cases where $V = x/t << 1/2$, $V >> 1/2$, and $V = 1/2$, alternative expressions for the waveheight may be developed. The effect of the simple pole at $\sigma = 1$ is either negligible or an easily evaluated residue contribution in these cases. These alternative expansions for $\eta(x/t)$ have intrinsic merit and in addition provide a check on the uniformly valid expression (35). Equations (6), (10) and (24) may be combined and rewritten as follows:

$$\eta(x,t) = I_m \left\{ \frac{i\pi}{2} e^{i(x-t)} H(1-2x/t) - \int_{C_2} + O(t^{-3}) \right\} \quad (V \neq 1/2)$$

The asymptotic behavior of $\eta(x,t)$ can be evaluated for $V << 1/2$ and $V >> 1/2$ by investigating the behavior of $\int_{C_2}$. This integral (after a suitable change in variable) can be expressed as in (12). Laurent expansions of the integrand of (12) subject to the restrictions on V above have been developed in section 3 of Appendix D. Substitution of these expansions into (12) yields the following result:

$$\int_{C_2} \frac{\sigma^2}{\sigma^4 - 1} e^{it(\sigma^2 V - \sigma)} d\sigma \sim 4V^{3/2} e^{-i\left(\frac{t}{4V} - \frac{\pi}{4}\right)} \int_0^\infty \left[\frac{\tau^{1/2}}{(1-16V^4)} + \right.$$

$$\left. + \frac{4iV\{3+12(4V^2)^2+(4V^2)^4\}}{(1 - 16V^4)^3} \tau^{1/2} + O(\tau^{3/2}) \right] e^{-t\tau} d\tau$$

Integrating term by term, the following asymptotic expansion for $\int_{C_2}$ is obtained:

$$\int_{C_2} \sim 4V^{3/2}\, e^{-i\left(\frac{t}{4V} - \frac{\pi}{4}\right)} \left[\frac{\Gamma(\frac{1}{2})\, t^{-1/2}}{(1-16V^4)} + \right.$$

$$\left. \frac{4iV\{3+12(4V^2)^2+(4V^2)^4\}}{(1-16V^4)^3} \Gamma(\tfrac{3}{2})\, t^{-3/2} + 0(t^{-5/2}) \right] \qquad (V>>1/2, V<<1/2)$$

The waveheight $\eta(x,t)$ can now be written:

$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{2}\, e^{i(x-t)}\, H(1-2x/t) + \right.$$

$$\left. + (-4)V^{3/2}\, e^{-i\left(\frac{t}{4V} - \frac{\pi}{4}\right)} \left[\frac{\Gamma(\frac{1}{2}, t^{-1/2}}{(1-16V^4)} + 0(t^{-3/2}) \right] \right\} \qquad (V \neq 1/2)$$

For the case $V = x/t = 1/2$, the Laurent expansions given in section 4, Appendix D, apply. The simple pole at $\sigma = 1$ is now coincident with the saddle point at $\sigma = 1/2V$. Therefore, the contour C_2 must be indented and the residue contribution accounted for. The waveheight $\eta(x,t)$ is given by:

$$\eta(x,t) = I_m \left\{ \frac{i\pi}{4}\, e^{i(x-t)} - \int_{C_2} + 0(t^{-3}) \right\} \qquad (V=1/2)$$

Substitution of the appropriate Laurent expansions into the integrand of $\int_{C_2}$ yields:

$$\int_{C_2} \sim e^{-i(\frac{t}{2} - \frac{\pi}{4})} \int_0^\infty \left[\frac{\sqrt{2}}{8}\tau^{-1/2} + \frac{3\sqrt{2}i}{16}\tau^{1/2} + \frac{9\sqrt{2}}{32}\tau^{3/2} + 0(\tau^{5/2})\right] e^{-t\tau}d\tau$$

$$\sim e^{-i(\frac{t}{2} - \frac{\pi}{4})}\left[\frac{\sqrt{2}}{8}\Gamma(\tfrac{1}{2})\,t^{-1/2} + \frac{3\sqrt{2}i}{16}\Gamma(\tfrac{3}{2})\,t^{-3/2} + \frac{9\sqrt{2}}{32}\Gamma(\tfrac{5}{2})\,t^{-5/2} + 0(t^{-7/2})\right]$$

The waveheight $\eta(x,t)$ can now be expressed as:

$$\eta(x,t) \sim I_m \left\{ \frac{i\pi}{4}\, e^{i(x-t)} + \right.$$

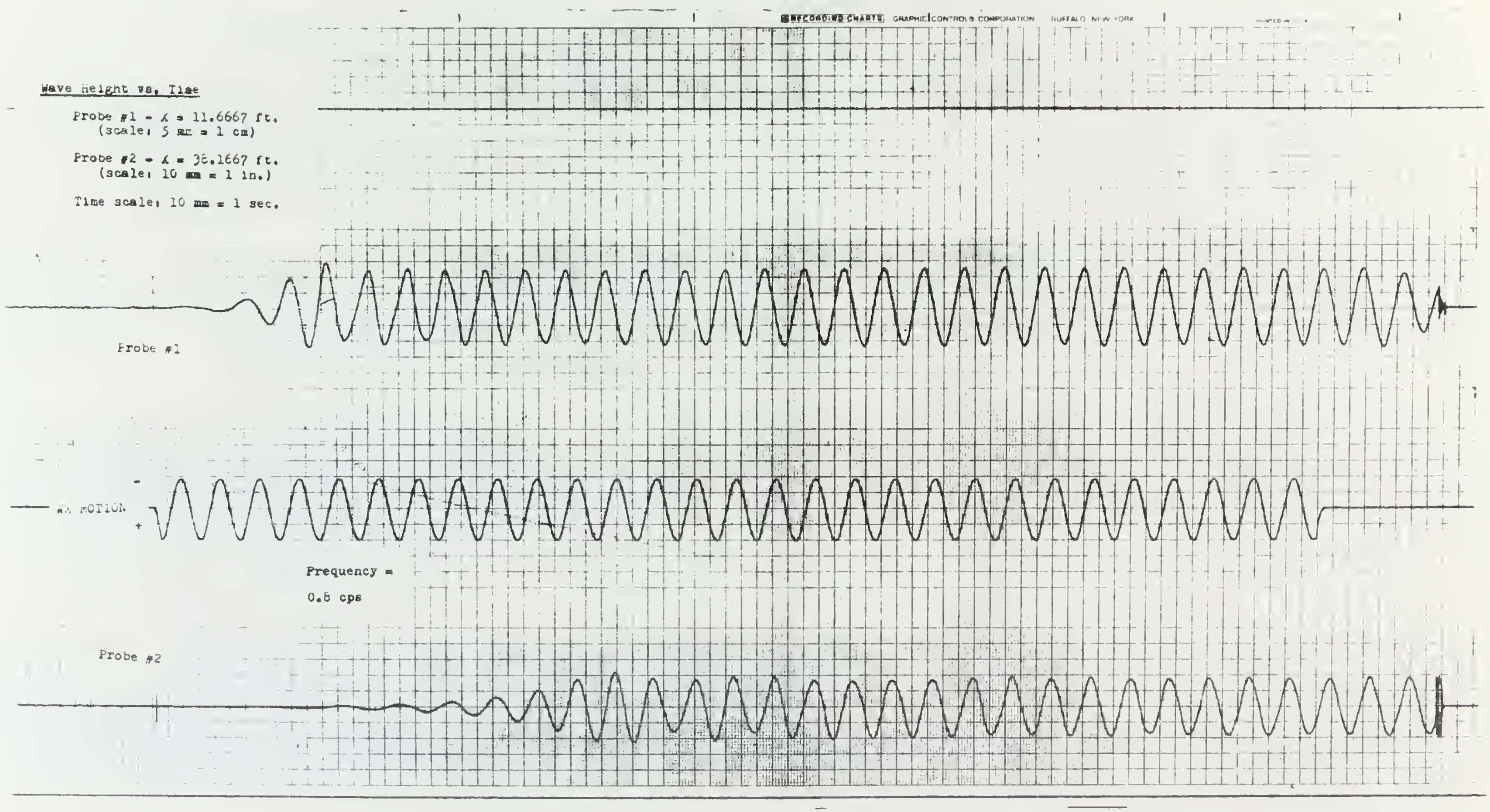
$$\left. + (-)e^{-i(\frac{t}{2} - \frac{\pi}{4})} \left[\frac{\sqrt{2}}{8}\,\Gamma(\tfrac{1}{2})\ t^{-1/2} + 0(t^{-3/2}) \right] \right. \qquad (V = 1/2)$$

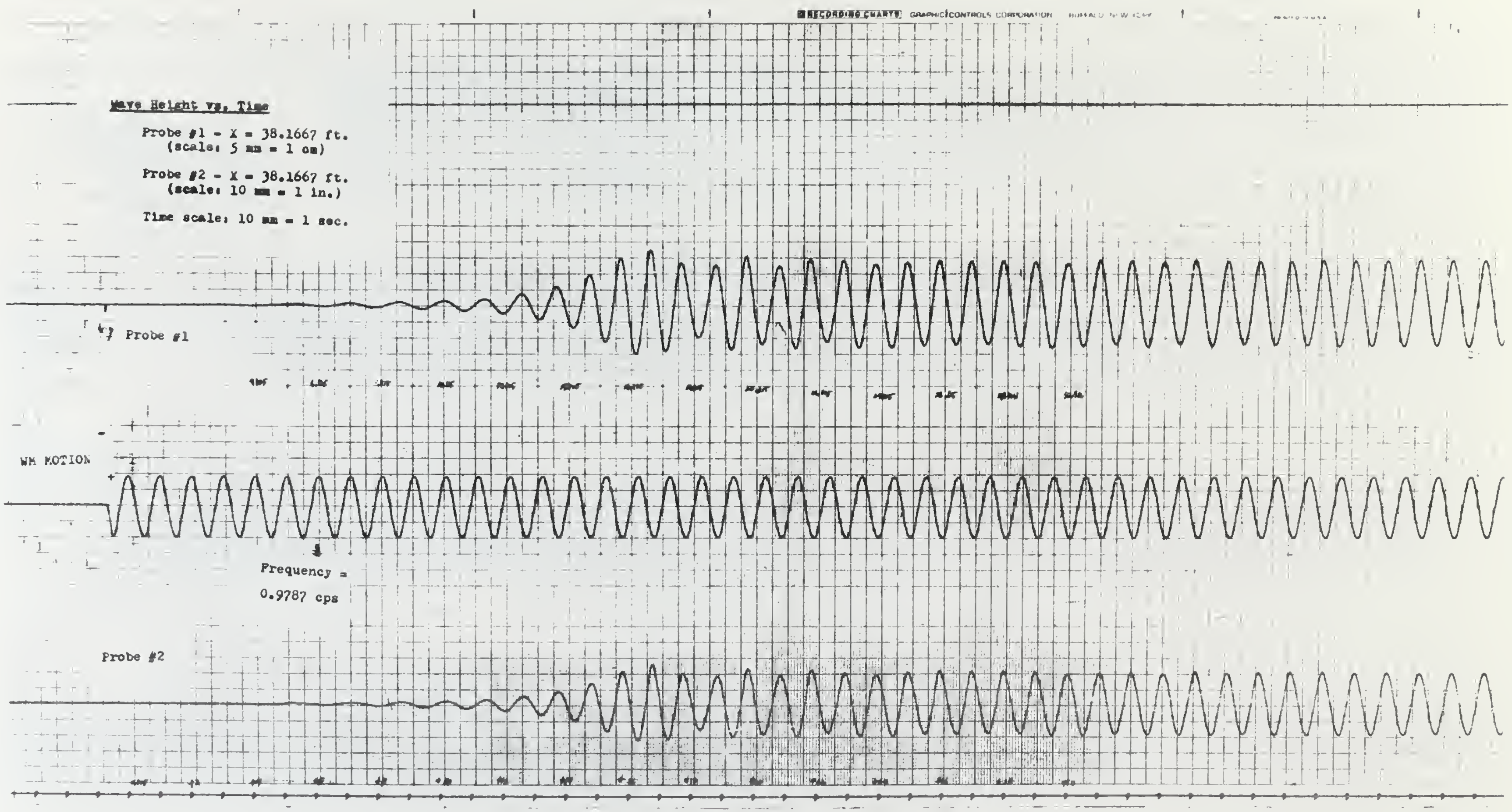
APPENDIX F

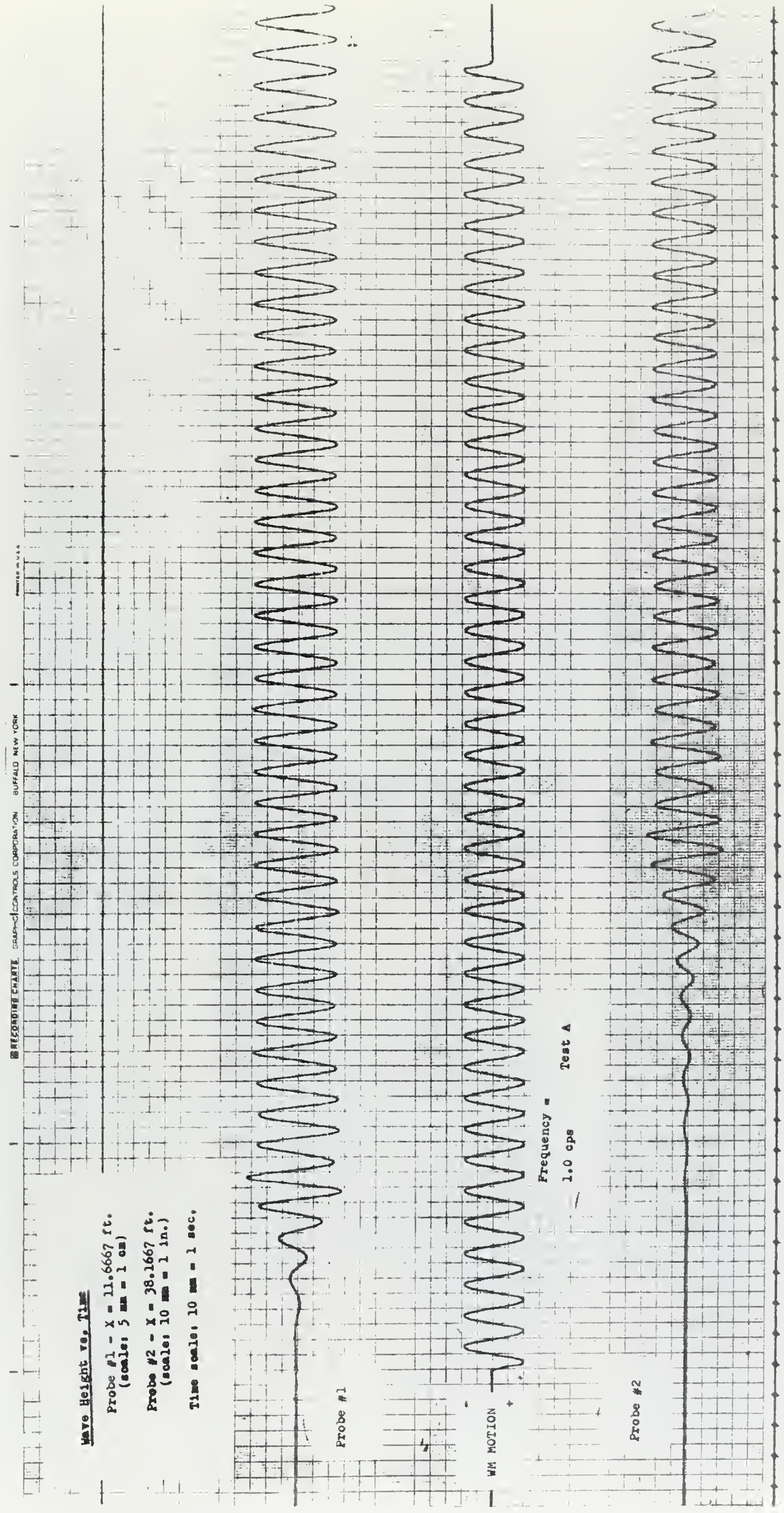
EXPERIMENTAL DATA

During the experiments continuous readings of wave height and time were made at two locations in the Ship Model Towing Tank with the use of a Sanborn, Series 7700, strip chart recorder. The motions of the wavemaker paddle were also continuously monitored and recorded simultaneously with the wave data. Time was measured from the instant of initial motion of the wavemaker paddle.

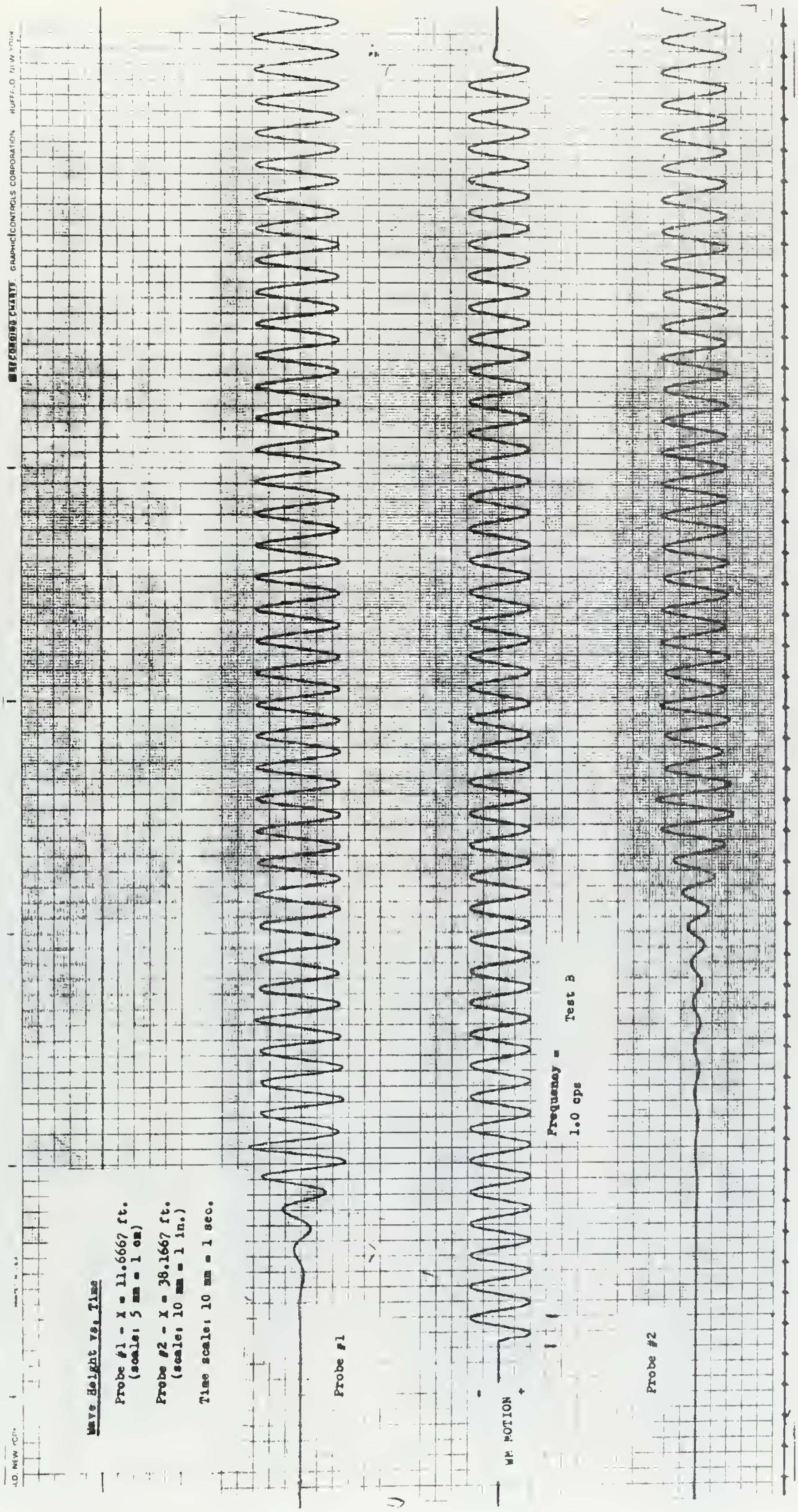
Reproductions of the wave records obtained as the output of the strip chart recorder are presented in order of increasing wavemaker frequency. Time is measured horizontally and wave height vertically according to the scales indicated.

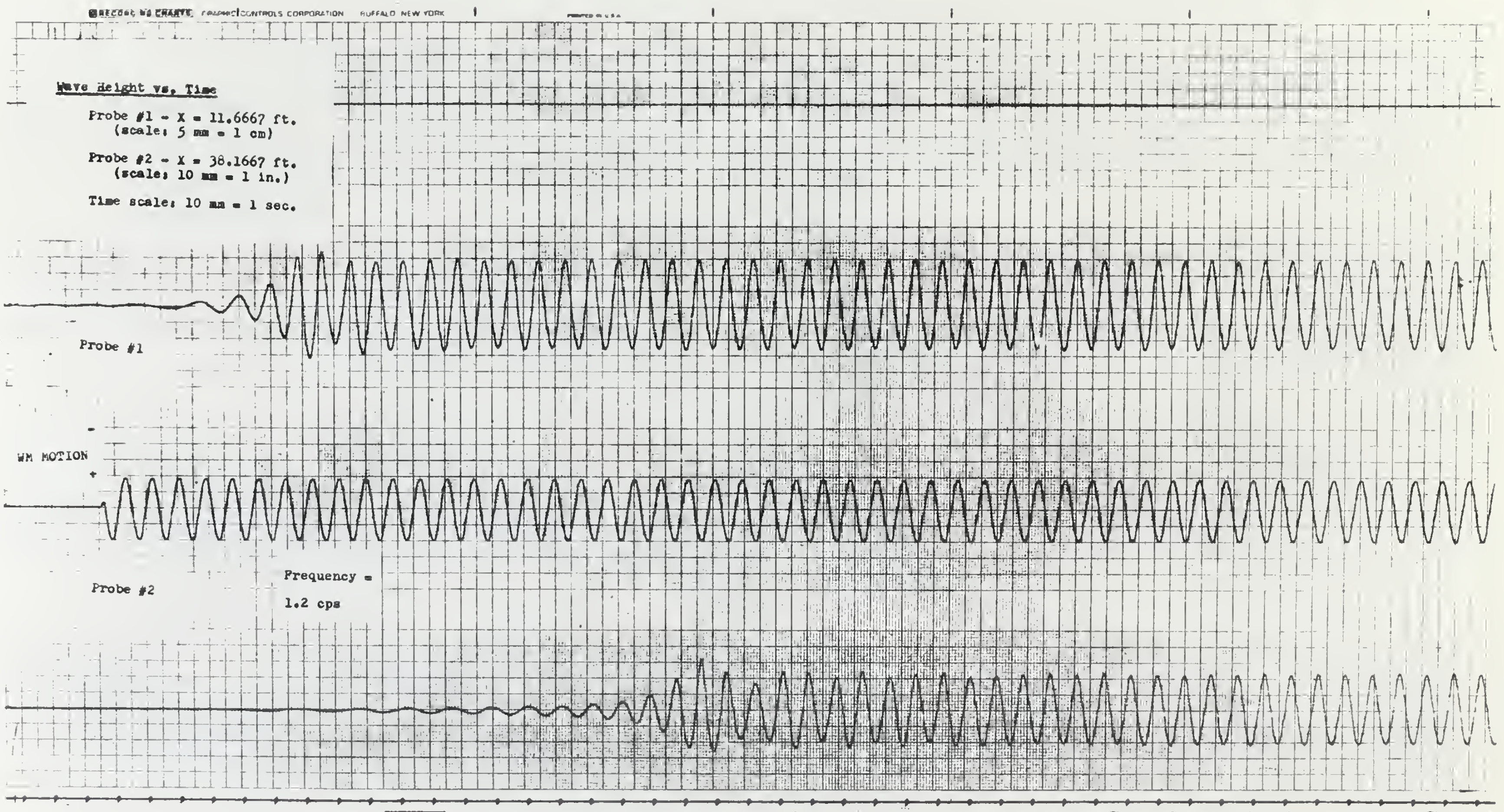

Wave Height vs. Time
Probe #1 - λ = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - λ = 36.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
RECORDING CHARTS GRAPHIC CONTROLS CORPORATION BUFFALO NEW YORK
Probe #1
MOTION
Frequency =
0.5 cps
Probe #2

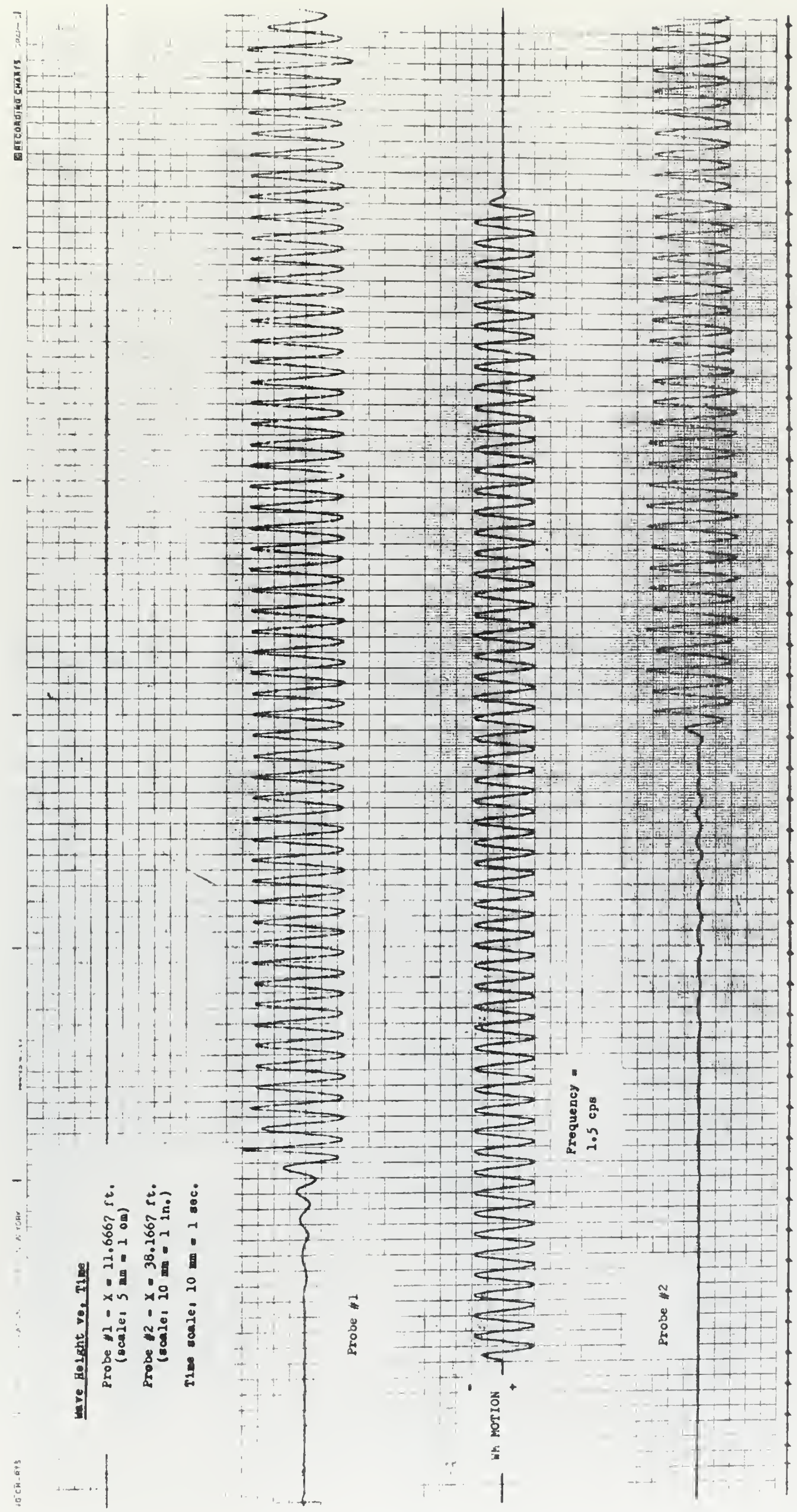

RECORDING CHARTS GRAPHIC CONTROLS CORPORATION
Wave Height vs. Time
Probe #1 - X = 38.1667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Frequency =
0.9787 cps
Probe #2

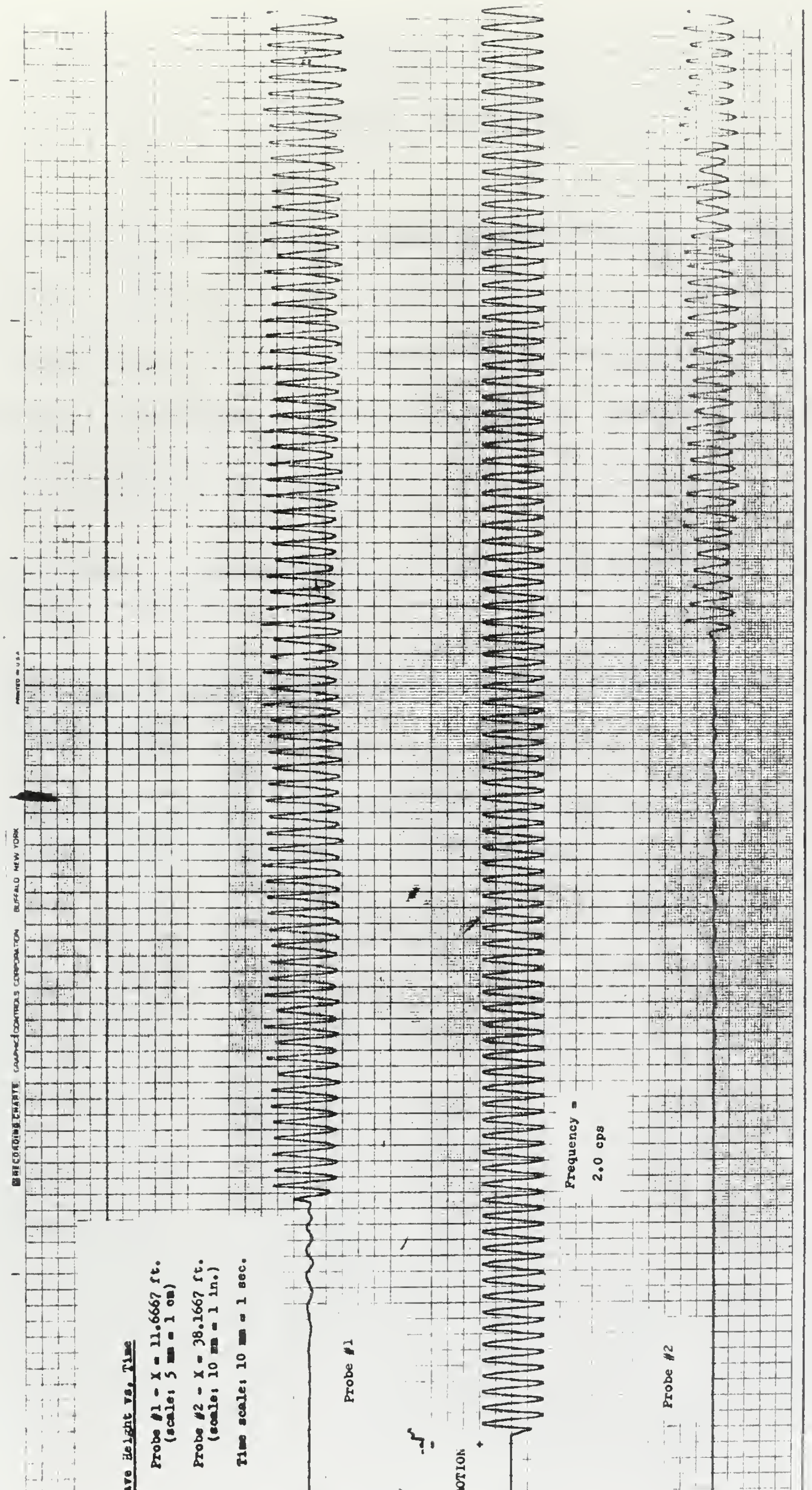


Wave Height vs. Time
Probe #1 - X = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Frequency = 1.0 cps
Test A
Probe #2
RECORDING CHARTS GRAPHIC CONTROLS CORPORATION BUFFALO NEW YORK


Wave Height vs. Time
Probe #1 - X = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Frequency = 1.0 cps
Test B
Probe #2


RECORD CHARTS GRAPHIC CONTROLS CORPORATION BUFFALO NEW YORK
Wave Height vs. Time
Probe #1 - X = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Probe #2
Frequency =
1.2 cps


Wave Height vs. Time
Probe #1 - X = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Frequency =
1.5 cps
Probe #2


Wave Height vs. Time
Probe #1 - X = 11.6667 ft.
(scale: 5 mm = 1 cm)
Probe #2 - X = 38.1667 ft.
(scale: 10 mm = 1 in.)
Time scale: 10 mm = 1 sec.
Probe #1
WM MOTION
Frequency =
2.0 cps
Probe #2
RECORDING CHARTS GRAPHIC CONTROLS CORPORATION BUFFALO NEW YORK

APPENDIX G

COMPUTER PROGRAMS

The facilities of the Computation Center of the Massachusetts Institute of Technology were used extensively in the comparison of theoretical to experimental results. The following programs were developed to calculate and plot the waves predicted by equation (35) for $\eta(x,t)$:

1) Timewise distribution of wave height for fixed distance.

2) Spacewise distribution of wave height for fixed time.

3) Function subprogram for the error function complement for complex arguments.

```
C     ANALYTICAL EVALUATION OF WAVEHT BASED ON EQUATION (35)                WAVE0001
C     THIS PROGRAM DEVELOPES THE TIMEWISE DISTRIBUTION OF WAVE HEIGHT.      WAVE0002
C     THE WAVES SEEN BY AN OBSERVER LOCATED AT A FIXED POSITION FROM THE WAVEMA- WAVE0003
C     KER WITH INCREASING TIME ARE PLOTTED BY THE SC-4020 PLOTTER.          WAVE0004
      COMPLEX C,D,CMPLX,CEXP,CERFC                                          WAVE0005
      DIMENSION X(1002),T(1002),ETA1(1002),JJJ(10)                          WAVE0006
      CALL STOIDV('M7950-8729',9,0)                                         WAVE0007
      CALL CAMRAV(9)                                                        WAVE0008
      CALL SETMIV(94,115,300,300)                                           WAVE0009
C     II IS THE TOTAL NUMBER OF DISTANCES X AT WHICH THE TIMEWISE DISTRIBUTION WAVE0010
C     IS TO BE DEVELOPED.                                                   WAVE0011
      READ(5,100)II                                                         WAVE0012
  100 FORMAT(I4)                                                            WAVE0013
      READ(5,101)(X(I),I=1,II)                                              WAVE0014
C     X = DISTANCE FROM THE WAVEMAKER.                                      WAVE0015
  101 FORMAT(10F8.0)                                                        WAVE0016
C     JJJ IS THE NUMBER OF POINTS TO BE PLOTTED BY THE SC-4020.             WAVE0017
      READ(5,102)(JJJ(I),I=1,II)                                            WAVE0018
  102 FORMAT(10I4)                                                          WAVE0019
C     FOR THIS RUN THE WRITE STATEMENT HAS BEEN REMOVED.                    WAVE0020
      IADV=2                                                                WAVE0021
      PI=3.1415927                                                          WAVE0022
      Q=1.                                                                  WAVE0023
      R=2.                                                                  WAVE0024
      S=4.                                                                  WAVE0025
      DO 20 I=1,II                                                          WAVE0026
      JJ=JJJ(I)                                                             WAVE0027
      F=0.                                                                  WAVE0028
      DO 19 L=1,JJ                                                          WAVE0029
      T(L)=F+0.5                                                            WAVE0030
      F=F+0.25                                                              WAVE0031
   19 CONTINUE                                                              WAVE0032
      DO 21 J=1,JJ                                                          WAVE0033
      V=X(I)/T(J)                                                           WAVE0034
      A=SQRT(PI*V)*(Q-S*V-S*V**2)/(R*SQRT(T(J))*(Q+R*V)*(Q+S*V**2))         WAVE0035
      B=V**1.5*((Q+S*V**2)**3-S*V*(Q+R*V)**3*(3.-S*V**2))*SQRT(PI)/(T(J)    WAVE0036
```

```
     1**1.5*(Q+R*V)**3*(Q+S*V**2)**3)                                   WAVE0037
      C=CMPLX(A,B)                                                      WAVE0038
      D=CEXP((0.,1.)*(T(J)/(S*V)-PI/S))                                 WAVE0039
      ETA=AIMAG((0.,1.)*PI*CEXP((0.,1.)*T(J)*(V-Q))*CERFC(SQRT(X(I))*(Q-WAVE0040
     1Q/(R*V))*0.707107*(1.,1.))/S+C*D)                                 WAVE0041
C      WRITE STATEMENT FOR PRINTOUT OF ETA, X, AND T MAY BE INSERTED HERE.  WAVE0042
      ETA1(J)=ETA                                                       WAVE0043
 21   CONTINUE                                                          WAVE0044
      IF(I.EQ.6) GO TO 17                                               WAVE0045
      AA=200.0                                                          WAVE0046
      GO TO 18                                                          WAVE0047
 17   AA=250.0                                                          WAVE0048
 18   CALL GRID1V(IADV,0.0,AA,-3.0,3.0,5.0,0.1,4,5,-4,-5,4,3)           WAVE0049
C     GRAPH IS TRACED TWICE FOR ADDED CANTRAST.                         WAVE0050
      CALL GRAF1V(T,ETA1,IERR,JJ,1)                                     WAVE0051
      CALL GRAF1V(T,ETA1,IERR,JJ,1)                                     WAVE0052
      GO TO (32,32,20,20),IADV                                          WAVE0053
 32   IADV=IADV+2                                                       WAVE0054
 20   CONTINUE                                                          WAVE0055
      CALL PLTND(N)                                                     WAVE0056
      WRITE(6,120)N                                                     WAVE0057
 120  FORMAT(' NO OF FRAMES IS ',I5)                                    WAVE0058
      CALL EXIT                                                         WAVE0059
      END                                                               WAVE0060
```

```
C     ANALYTICAL EVALUATION OF WAVEHT BASED ON EQUATION (35)                    WAVE0001
C     THIS PROGRAM DEVELOPES THE SPACEWISE DISTRIBUTION OF WAVE HEIGHT.         WAVE0002
C     WAVE HEIGHT IS PLOTTED VERSUS DISTANCE FOR FIXED TIME BY THE SC-4020.     WAVE0003
      COMPLEX C,D,CMPLX,CEXP,CERFC                                              WAVE0004
      DIMENSION X(802),T(802),ETA1(802)                                         WAVE0005
      CALL STOIDV('M7950-8729',9,0)                                             WAVE0006
      CALL CAMRAV(9)                                                            WAVE0007
      CALL SETMIV(94,115,300,300)                                               WAVE0008
C     II IS THE NUMBER OF TIMES T AT WHICH THE SPACEWISE DISTRIBUTION IS TO BE  WAVE0009
C     DETERMINED.                                                               WAVE0010
C     JJ IS THE NUMBER OF POINTS TO BE PLOTTED BY THE SC-4020.                  WAVE0011
      READ(5,100)II,JJ                                                          WAVE0012
C     T = ELAPSED TIME SINCE INITIAL MOTION OF THE WAVEMAKER.                   WAVE0013
      READ(5,101)(T(I),I=1,II)                                                  WAVE0014
  100 FORMAT(2I4)                                                               WAVE0015
  101 FORMAT(10F8.0)                                                            WAVE0016
C     FOR THIS RUN THE WRITE STATEMENT HAS BEEN REMOVED.                        WAVE0017
      IADV=2                                                                    WAVE0018
      PI=3.1415927                                                              WAVE0019
      Q=1.                                                                      WAVE0020
      R=2.                                                                      WAVE0021
      S=4.                                                                      WAVE0022
      F=0.                                                                      WAVE0023
      DO 19 L=1,JJ                                                              WAVE0024
      X(L)=F+0.5                                                                WAVE0025
      F=F+0.25                                                                  WAVE0026
   19 CONTINUE                                                                  WAVE0027
      DO 21 J=1,II                                                              WAVE0028
      DO 20 I=1,JJ                                                              WAVE0029
      V=X(I)/T(J)                                                               WAVE0030
      A=SQRT(PI*V)*(Q-S*V-S*V**2)/(R*SQRT(T(J))*(Q+R*V)*(Q+S*V**2))             WAVE0031
      B=V**1.5*((Q+S*V**2)**3-S*V*(Q+R*V)**3*(3.-S*V**2))*SQRT(PI)/(T(J)        WAVE0032
     1**1.5*(Q+R*V)**3*(Q+S*V**2)**3)                                           WAVE0033
      C=CMPLX(A,B)                                                              WAVE0034
      D=CEXP((0.,1.)*(T(J)/(S*V)-PI/S))                                         WAVE0035
      ETA=AIMAG((0.,1.)*PI*CEXP((0.,1.)*T(J)*(V-Q))*CERFC(SQRT(X(I))*(Q-        WAVE0036
```

```
     1O/(R*V))*0.707107*(1.,1.))/S+C*D)                                 WAVE0037
C      WRITE STATEMENT FOR PRINTOUT OF ETA, X, AND T MAY BE INSERTED HERE.   WAVE0038
      ETA1(I)=ETA                                                       WAVE0039
 20   CONTINUE                                                          WAVE0040
      CALL GRID1V(IADV,0.0,200.0,-3.0,3.0,5.0,0.1,4,5,-4,-5,4,3)        WAVE0041
C     GRAPH IS TRACED TWICE FOR ADDED CANTRAST.                         WAVE0042
      CALL GRAF1V(X,ETA1,IERR,800,1)                                    WAVE0043
      CALL GRAF1V(X,ETA1,IERR,800,1)                                    WAVE0044
      GO TO (32,32,21,21),IADV                                          WAVE0045
 32   IADV=IADV+2                                                       WAVE0046
 21   CONTINUE                                                          WAVE0047
      CALL PLTND(N)                                                     WAVE0048
      WRITE(6,120)N                                                     WAVE0049
 120  FORMAT(' NO OF FRAMES IS ',I5)                                    WAVE0050
      CALL EXIT                                                         WAVE0051
      END                                                               WAVE0052
```

```
      COMPLEX FUNCTION CERFC(Z)
C     THIS FUNCTION SUBPROGRAM DETERMINES VALUES OF THE ERROR FUNCTION COMPLE-
C     MENT FOR COMPLEX ARGUEMENTS.
C     AN INFINITE SERIES APPROXIMATION TO THE ERFC IS USED.
      COMPLEX A,B,F,G,H,Z,CEXP
      PI=3.1415927
      Q=1.
      R=2.
      S=4.
      X=REAL(Z)
      Y=AIMAG(Z)
      IF(ABS(X).LE.0.0001) GO TO 1
      A=ERF(X)+EXP(-X**2)*((Q-COS(R*X*Y))+(0.,1.)*SIN(R*X*Y))/(R*PI*X)
      GC TO 2
 1    A=ERF(X)+EXP(-X**2)*(R*Y*SIN(R*X*Y)-(0.,1.)*R*Y*COS(R*X*Y))/(R*PI)
     1-R*X*EXP(-X**2)*((Q-COS(R*X*Y))+(0.,1.)*SIN(R*X*Y))/(R*PI)
 2    B=0.
      N=1
 9    IF(ABS(N*Y).GE.174.0.OR.N**2/S.GE.174.0) GO TC 3
      F=R*X-R*X*COSH(N*Y)*COS(R*X*Y)+N*SINH(N*Y)*SIN(R*X*Y)
      G=R*X*COSH(N*Y)*SIN(R*X*Y)+N*SINH(N*Y)*COS(R*X*Y)
      H=R*EXP(-(X**2))*EXP(-(N**2)/S)*(F+(0.,1.)*G)/(PI*(N**2+S*X**2))
      B=B+H
      N=N+1
      C=CABS(H)
      IF(C.GT.0.00001) GO TO 9
 3    CERFC=(1.,0.)-A-B
      RETURN
      END
```

APPENDIX H

NUMERICAL ANALYSIS OF $\eta(x,t)$

1. Short Time Solution.

The asymptotic behavior of equation (5) for the wave-height $\eta(x,t)$ is readily evaluated for large values of t (or x). This behavior is described by equation (35). There appears to be no such expression for small values of t (or x). For this case numerical methods must be used with the aid of computational facilities a practical necessity. For very small values of t, equation (2) may be evaluated in its present form - integrating numerically; e.g., using Simpson's rule. As t (or x) grows in magnitude, however, this approach becomes unsatisfactory due to the increasingly oscillatory nature of the integrand of (2).

2. Programming Difficulties and the Solution.

The numerical integration difficulties posed by the highly oscillatory nature of the integrand of (2) can be avoided by employing the contour integration methods and appropriate changes of variable used in determing the asymptotic behavior of equation (5). Here the integrals I_1, I_2, I_3 and I_4 (defined by (5)) are regarded as contour integrals in the complex $\underset{\sim}{\sigma}$ plane and equivalent contours which facilitate their evaluation have been determined.

Integrals I_1, I_3 and I_4 can therefore be represented by (see Appendix C):

$$I_1 = i\pi\text{Res}(1) + \int_{C_{SD}} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2V+\sigma)} d\sigma$$

$$I_3 = i\pi\text{Res}(1) + e^{it} \int_{C_{SD}} \frac{\sigma}{\sigma^4-1} e^{i\sigma^2x} d\sigma$$

$$I_4 = i\pi\text{Res}(1) + e^{-it} \int_{C_{SD}} \frac{\sigma}{\sigma^4-1} e^{i\sigma^2x} d\sigma$$

Introducing the change of variable $i\tau_1 = \sigma^2V+\sigma$ into I_1 and $i\tau_2 = \sigma^2$ into I_3 and I_4, these integrals can be written:

$$I_1 = i\pi\text{Res}(1) + \int_0^\infty \frac{\sigma^2}{\sigma^4-1} \frac{d\sigma}{d\tau_1} e^{-t\tau_1} d\tau_1$$

$$I_3 = i\pi\text{Res}(1) + e^{it}\int_0^\infty \frac{\sigma}{\sigma^4-1} \frac{d\sigma}{d\tau_2} e^{-x\tau_2} d\tau_2$$

$$I_4 = i\pi\text{Res}(1) + e^{-it}\int_0^\infty \frac{\sigma}{\sigma^4-1} \frac{d\sigma}{d\tau_2} e^{-x\tau_2} d\tau_2$$

When expressed in these forms, the oscillatory nature of the integrand has been removed and in fact it now becomes exponentially small as τ_1 and τ_2 become large. Numerical

evaluation of these integrals can now be accomplished by a variety of methods for small and large t (or x).

A similar method can be used to evaluate I_2. Its representation is given by equations (9) and (23) for the cases $V = x/t < 1/2$ and $V > 1/2$. Considering only equation (23), I_2 can be written:

$$I_2 = i\pi \mathrm{Res}(1) + \int_{C_1} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2 V-\sigma)} d\sigma + \int_{C_2} \frac{\sigma^2}{\sigma^4-1} e^{it(\sigma^2 V-\sigma)} d\sigma \qquad (V > 1/2)$$

where C_1 is the steepest descent path from the origin to infinity and C_2 includes both steepest descent paths from the saddle point to infinity. Introducing the change in variable $i\tau_1 = \sigma^2 V-\sigma$ into $\int_{C_1}$ and $-1/4V + i\tau_2 = \sigma^2 V-\sigma$ into $\int_{C_2}$, I_2 can be expressed as:

$$I_2 = i\pi \mathrm{Res}(1) + \int_0^\infty \frac{\sigma^2}{\sigma^4-1} \frac{d\sigma}{d\tau_1} e^{-t\tau_1} d\tau_1 +$$

$$+ e^{-it/4V} \int_0^\infty \frac{\sigma^2}{\sigma^4-1} \frac{d\sigma}{d\tau_2} e^{-t\tau_2} d\tau_2 \qquad (V > 1/2)$$

A similar result can be obtained from equation (9) and the case $V < 1/2$ - the only difference being a change in sign of

the residue term. Numerical evaluation of I_2 is now readily accomplished. Finally, equation (5) which relates integrals I_1, I_2, I_3 and I_4 to the waveheight, is used to complete the numerical evaluation of $\eta(x,t)$.

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into calm water.